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Review of Engineering Dynamics

Part 1: Kinematics of Particles and Rigid Bodies

by

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Contents

1.0 Course Overview

Many engineers deal with the analysis and design of systems that are in motion. Determining the behavior of systems in motion requires knowledge of engineering dynamics, which is a branch of engineering mechanics dealing with the motion of bodies under the action of forces. It is critical to understand forces and how they change during motion to design parts for maximum forces.

This course focuses on the most essential concepts of engineering dynamics and will outline a systematic approach to solving dynamics problems. Following the step-by-step process presented in this course will help you to quickly determine the appropriate equations to use for problems relating to engineering dynamics. The key is being able to determine what type of problem you are trying to solve, then you can determine which equation to use. The basic idea of this procedure is outlined in Section 2.6.

This complete course is divided into two parts: part 1 focuses on kinematics of particles and rigid bodies and part 2 focuses on kinetics of particles and rigid bodies. The course is intended as a review of dynamics, so a previous knowledge of the subject is helpful. The course, however, can be treated as a fundamental introduction to the topics.

2.0 Basic Introductory Concepts

2.1 Introduction

Before starting with the solution procedures for specific problem categories, it is critical to understand some basic introductory concepts. This section will define some basic terminology and classify the different types of energy along with the appropriate equations for each type. You must understand the differences between kinematics and kinetics, as well as the differences between rigid bodies and particles. Therefore, this introductory section will also define and describe those terms.

2.1.1 Vectors vs. Scalars

Vector quantities have both magnitude and direction. Scalar quantities only have magnitude. Throughout this course, vector quantities will be designated with an arrow above the variable (such as velocity $\vec{v}$). Keep in mind that different sources use different notation for vectors. Bold text, for example, is also a common way to denote a vector quantity. Table 1 gives examples of common vectors and scalars.

Table 1 Examples of vectors and scalars

	Definition	Examples
Vector	Magnitude and direction	Momentum, velocity, acceleration, force
Scalar	Magnitude only	Potential energy, kinetic energy, time

It is assumed that the reader has a familiarity with concepts of vector operations such as vector addition and unit vectors. Brief summaries of these operations will be provided where necessary, but readers are encouraged to review these topics in more detail if required.

2.2 Planar Motion

Dynamic motion can occur in two or three dimensions. Two-dimensional motion occurs in a single plane and is very common in engineering applications. This course will only focus on two-dimensional planar motion. Concepts of three-dimensional motion become more complex in the vector operations, but the basic principles are the same as planar motion.

2.3 Types of Energy

A lot of dynamics problems focus on energy and how energy changes form during the motion. Two common methods utilizing energy are Work and Energy and Conservation of Energy, and both of these will be discussed more in part 2 of this course. You need to understand the different forms of energy to be able to solve many engineering dynamics problems.

2.3.1 Potential Energy

Potential energy is energy at rest. There are a couple of different forms of potential energy that are common in dynamics problems: gravitational potential energy and elastic potential energy. The first form is gravitational potential energy, which is the stored energy due to an elevated position (water stored in an elevated reservoir has gravitational potential energy). Gravitational potential energy for a mass (*m*) at a height (*h*) above the datum is shown in Equation 1.

Gravitational Potential Energy	$V = mgh$	Equation 1

Note that the gravitational potential energy does not depend on the path taken by the object, but only on its elevation above a user defined datum.

Elastic potential energy is a stored energy of a spring or other elastic member. The potential energy of a stretched spring with constant (*k*) deformed a distance (*x*) from its equilibrium position is given in Equation 2.

Elastic Potential Energy	$V_e = \frac{1}{2}kx^2$	Equation 2

2.3.2 Kinetic Energy

Kinetic energy is energy related to motion. The general equation for kinetic energy for a mass (*m*) moving at a velocity (*v*) is given in Equation 3.

$$T = \frac{1}{2}mv^2 \qquad \text{Equation 3}$$

2.4 Kinematics and Kinetics

One very important area in understanding dynamics is the distinction between kinematics and kinetics. The determining factor here is force. Kinematics problems concentrate on motion without any regard to the force required for that motion. In other words, kinematics focusses on the geometry of motion. Part 1 of this course focuses on problems relating to kinematics.

Kinetics problems are focused on what force is required to cause a motion. It is important to have a good understanding of kinematics before moving into topics on kinetics. Problems in kinetics are covered in detail in part 2 of this course.

2.5 Particles vs. Rigid Bodies

It is also very important to understand the difference between particles and rigid bodies. It is common to hear the word particle and automatically assume something small. However, that is not necessarily the case. In dynamics problems, it is not uncommon to have something like a car or train be a particle. The defining difference between particles and rigid bodies is rotation. If the object is not rotating about its own centroid it can be treated as a particle because the object's rotational properties do not matter. If the object is rotating about its own centroid, the rotational properties are important and it is considered a rigid body.

2.6 Determination of the Problem Category

When you need to solve a dynamics problem the first step is to determine what type of problem you are trying to solve. Dynamics problems will always fall into one of two main groups: particle problems and rigid body problems. Each of the two main groups will then be subdivided into either kinematics problems or kinetics problems. Therefore, any problem you work in dynamics will fall into one of four major categories:

1. Particle kinematics (See Section 3.0)
2. Particle kinetics (See part 2 of this course)
3. Rigid body kinematics (See Section 4.0)
4. Rigid body kinetics (See Part 2 of this course)

The first step to solving a dynamics problem is to quickly determine which category the problem belongs in. Figure 1 illustrates the thought process. First, ask yourself if the system contains objects that are particles or rigid bodies. If rotational properties are not important, they are likely particles. Once you determine if you are working with particles or rigid bodies, determine if the question is asking about geometry of motion (kinematics) or forces (kinetics). After you know the proper problem category, you can go to that category section in this course and determine the proper steps for the solution process.

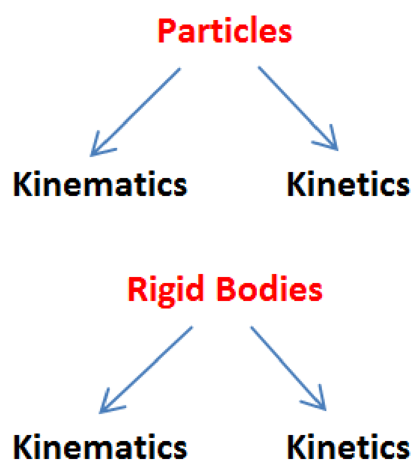


Figure 1 Categories of engineering dynamics problems

3.0 Kinematics of Particles

3.1 Introduction

The first group of problems we will examine is the kinematics of particles group, which is the easiest of all the groups. Motion of a particle can be described in many different ways, but the two main types of motion for particles will be rectilinear motion and curvilinear motion. Rectilinear motion, which will be covered in Section 3.2, is motion of a particle moving along a straight line. Curvilinear motion, which is covered in Section 3.3, is motion along a curved path. Both types of motion are illustrated in Figure 2.

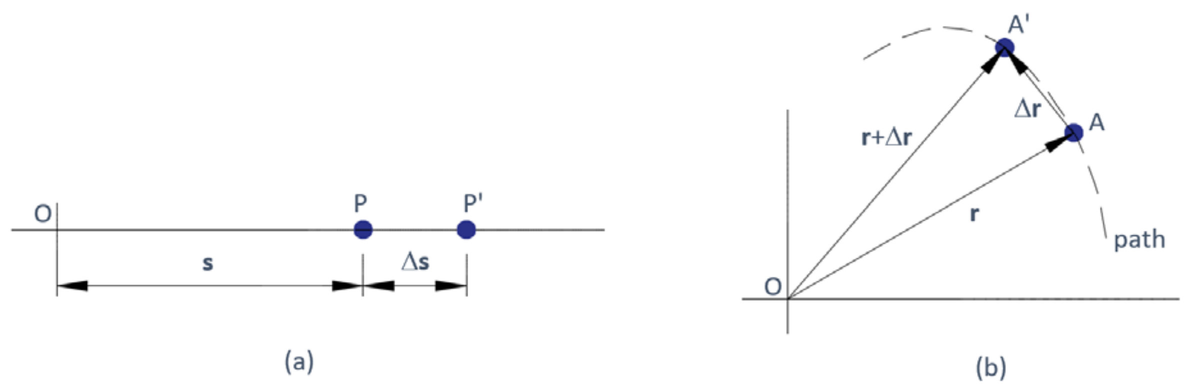


Figure 2 (a) Rectilinear motion and (b) Curvilinear motion

3.2 Rectilinear Motion

Rectilinear motion, as illustrated in Figure 2 (a), is motion of a particle along a straight line. When studying rectilinear motion of particles, it is common to determine the particle's position at various times during its motion. Velocity is the time rate of change of position and can be determined by dividing the change in position by the change in time. Instantaneous linear velocity is given by Equation 4, where s is position and t is time.

Linear Velocity $$v = \frac{ds}{dt}$$ Equation 4

The acceleration of a particle is determined by looking at how the velocity changes over time. The instantaneous acceleration is the rate of change of velocity with respect to time. As shown in Equation 5, acceleration is the first time derivative of velocity and the second time derivative of position.

Linear Acceleration $$a = \frac{d^2 s}{dt^2} = \frac{dv}{dt}$$ Equation 5

See Section 5.0 for a brief review of calculus if you need help understanding time derivatives.

Example 1

A particle moves along a straight line, and the motion of the particle is defined by the equation below (x and t are given in meters and seconds, respectively). Determine the velocity and acceleration of the particle at a time of 3 seconds.

$$x(t) = 3t^4 - 2t^3 + 4t - 5$$

Solution:

The velocity and acceleration equations are determined from differentiation.

$$v(t) = \frac{dx}{dt} = 12t^3 - 6t^2 + 4 \quad \left[\frac{m}{s}\right]$$

$$a(t) = \frac{dv}{dt} = 36t^2 - 12t \quad \left[\frac{m}{s^2}\right]$$

The velocity and acceleration of the particle at a time of 3 seconds is determined by substituting 3 in for time in the above equations.

$$v = 12(3)^3 - 6(3)^2 + 4 = 274\frac{m}{s}$$

$$a = 36(3)^2 - 12(3) = 288\frac{m}{s^2}$$

3.2.1 Constant Acceleration Motion

If acceleration is constant, the motion is classified as uniformly accelerated rectilinear motion. Many applications, such as free fall, are constant acceleration problems. For a particle in rectilinear motion with constant acceleration, the velocity at any time *t* will be given by Equation 6 and its position at any time *t* will be given by Equation 7.

Velocity $$v = v_0 + at$$ Equation 6

Position $$x = x_0 + v_0 t + \frac{1}{2}at^2$$ Equation 7

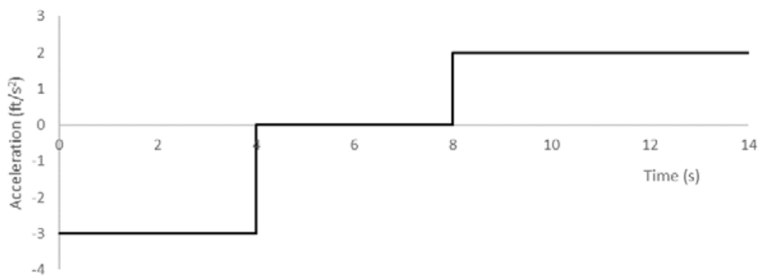
Another useful equation to calculate distance, velocity, or acceleration without knowing anything about time is given in Equation 8.

$$v^2 = v_0^2 + 2a(x - x_0)$$ Equation 8

Though the derivation of these equations has not been provided, the process is relatively straightforward and uses basic rules of calculus. The important thing to remember is that the equations above are only valid if the acceleration is constant.

Example 2

A particle moves along a straight line. The acceleration of the particle is shown in the graph below and the initial velocity is 6 ft/s. What is the particle velocity during the zero-acceleration period? What is the particle velocity at 14 seconds?



Solution:

This problem can be solved using equations or by graphical integration. The first part will be solved using the equations. From Equation 6, the velocity equation for the first four seconds will be

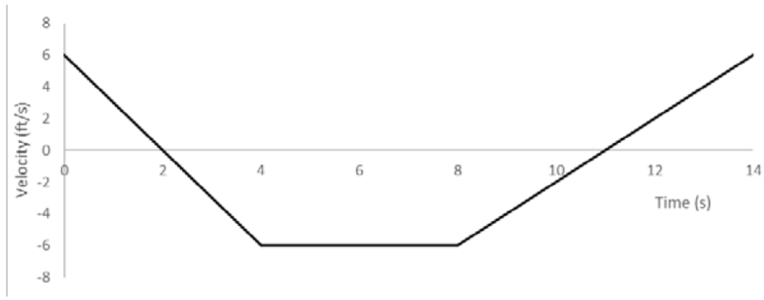
$$v = v_0 + at$$

$$v = 6 + (-3)t$$

This will result in a straight-line plot starting at 6 and having a slope of -3. At time equals four seconds

$$v = 6 + (-3)(4) = -6\frac{ft}{s}$$

The plot below shows the velocity, which will equal -6 ft/s at a time of four seconds. The velocity will remain constant during the zero-acceleration period.



The acceleration becomes 2 ft/s^2 at a time of 8 seconds, and it remains constant until 14 seconds. Using graphical integration, the change in velocity will equal the area under the curve. The velocity at 14 seconds will be

$$v_{14s} = -6\frac{ft}{s} + 2\frac{ft}{s^2}(14s - 8s) = 6\frac{ft}{s}$$

3.2.1.1 Free Fall

Free fall is a common example of a constant acceleration problem. If an object is in free fall, the constant acceleration is due to gravity.

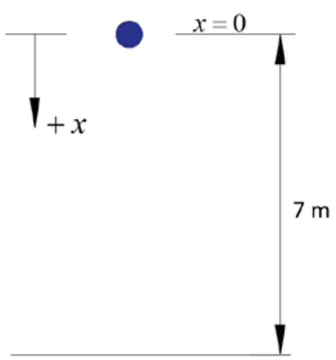
Example 3

An object is dropped from a distance of 7 meters above the ground. How long does it take to fall?

Solution:

This is a constant acceleration problem with acceleration being 9.8 m/s^2 downward (acceleration of gravity is $g = 9.81\frac{m}{s^2} = 32.2\frac{ft}{\sec^2}$). Position is given by Equation 7.

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$



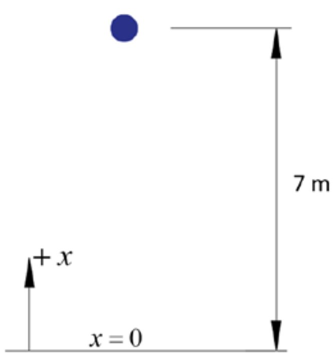
The initial position can be treated as zero, and the initial velocity is zero

$$x = \frac{1}{2} a t^2$$

Rearrange the equation to solve for time

$$t = \sqrt{\frac{2x}{a}} = \sqrt{\frac{2(7m)}{9.81 \, {}^{m}\!/_{s^2}}} = 1.19s$$

The problem could also be solved using the ground as the datum.



$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

$$x = 7m + 0 + \frac{1}{2}\left(-9.81 \frac{m}{s^2}\right) t^2$$

$$t = \sqrt{\frac{2x}{a}} = \sqrt{\frac{2(7m)}{9.81 \, {}^{m}\!/_{s^2}}} = 1.19s$$

3.2.2 Acceleration as a Function of Time

A more general approach is to have the acceleration of the particle as a function of time.

$$a = f(t) \qquad \text{Equation 9}$$

The constant acceleration motion discussed in Section 3.2.1 is a special case of Equation 9. Velocity and position are determined through integration.

$$v = v_0 + \int_0^t f(t)dt \qquad \text{Equation 10}$$

$$x = x_0 + \int_0^t v dt \qquad \text{Equation 11}$$

Again, refer to Section 5.0 for some general review of calculus if required. More specifically, Section 5.2 covers general rules of integration required to solve Equation 10 and Equation 11.

Example 4

A particle moving along a straight line has acceleration based on the function below (units of in/s^2). The initial position is zero inches and the initial velocity is 4 in/s. What is the function for velocity and position of this particle?

$$a(t) = 3t - 20$$

Solution:

The velocity function is found through integration.

$$v(t) = 4 + \int_0^t 3t - 20dt$$

$$v(t) = 4 + \frac{3}{2}t^2 - 20t \quad \left[\frac{in}{s}\right]$$

The position function if found by integrating the velocity function.

$$x(t) = x_0 + \int_0^t v dt$$

$$x(t) = \int_0^t \frac{3}{2}t^2 - 20t + 4 dt$$

$$x(t) = \frac{1}{2}t^3 - 10t^2 + 4t \quad [in]$$

3.3 Curvilinear Motion

3.3.1 Introduction

Curvilinear motion, as illustrated in Figure 2 (b), is motion of a particle along a curved path. Referring to the figure, the average velocity would be $\Delta r / \Delta t$. The general form of the velocity vector is determined by taking the limit as the time interval approaches zero, which gives the velocity shown in Equation 12.

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} \qquad \text{Equation 12}$$

Similarly, the acceleration is found by the derivative of velocity with respect to time.

$$\vec{a} = \frac{d\vec{v}}{dt} = \dot{\vec{v}} \qquad \text{Equation 13}$$

There are three coordinate systems commonly used for problems associated with curvilinear motion: rectangular coordinates, normal and tangential coordinates, and polar coordinates. The coordinate system best suited for a problem will depend on the type of motion.

3.3.2 Rectangular Coordinates

The first coordinate system we will discuss for curvilinear motion will be rectangular coordinates. Rectangular coordinates will use unit vectors, as shown in Figure 3.

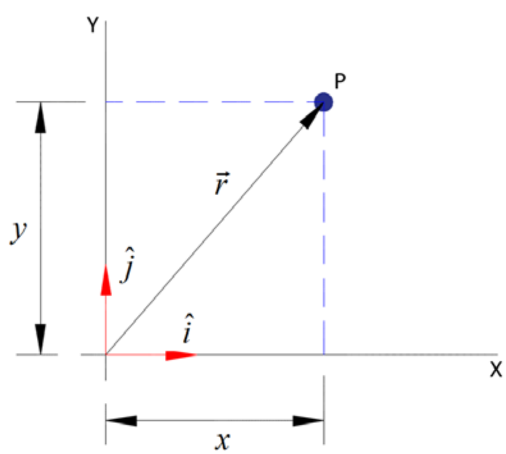


Figure 3 Rectangular coordinates

A unit vector is a vector that has a length of one unit. The unit vector along the x-axis is $\hat{i}$ and the unit vector along the y-axis is $\hat{j}$. A unit vector $\hat{k}$ also exists along the z-axis, though not shown in the figure. The position of a point P can be defined by a position vector $\vec{r}$ and can be expressed in terms of the unit vectors.

Position $$\vec{r} = x\hat{i} + y\hat{j}$$ Equation 14

Differentiating the position vector with respect to time will give velocity

Velocity $\vec{v} = \dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j}$ Equation 15

A second time derivate will give acceleration

Acceleration $\vec{a} = \ddot{\vec{r}} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$ Equation 16

Example 5

The position of a particle is defined by $\vec{r} = (3t^2)\hat{i} + (2t^4 - t + 1)\hat{j}$. Determine the equations for particle velocity and acceleration.

Solution:

The particle velocity is determined using Equation 15.

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} = (6t)\hat{i} + (8t^3 - 1)\hat{j}$$

The particle acceleration is determined using Equation 16.

$$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} = (6)\hat{i} + (24t^2)\hat{j}$$

3.3.2.1 Projectile Motion

Projectile motion is a common type of motion in the kinematics of particles group that utilizes rectangular coordinates. In projectile motion, the motion of the particle can be broken into the two components: horizontal and vertical motion. The horizontal motion will be constant velocity (if we assume no air resistance) and the vertical motion will be constant acceleration (acceleration due to gravity). The equations for the horizontal motion are given in Equation 17, and the equations for the vertical motion are shown in Equation 18. For all equations, the subscript 0 indicates initial conditions (at time = 0).

Horizontal Motion

$$\begin{aligned} a_x &= 0 \\ v_x &= (v_x)_0 \\ x &= x_0 + (v_x)_0 t \end{aligned}$$

Equation 17

Vertical Motion

$$\begin{aligned} a_y &= -g \\ v_y &= (v_y)_0 - gt \\ y &= y_0 + (v_y)_o t - \frac{1}{2} g t^2 \end{aligned}$$

Equation 18

3.3.3 Normal and Tangential Coordinates

Another method of defining curvilinear motion is by using tangential and normal coordinates. A common application of normal and tangential coordinates would be motion along a circular path. Figure 4 shows a particle P moving along a curved path. At the location of the particle, point C represents the center of curvature of the path. The velocity of the particle, as shown in Figure 4 (a), will be tangent to the path in the direction of motion.

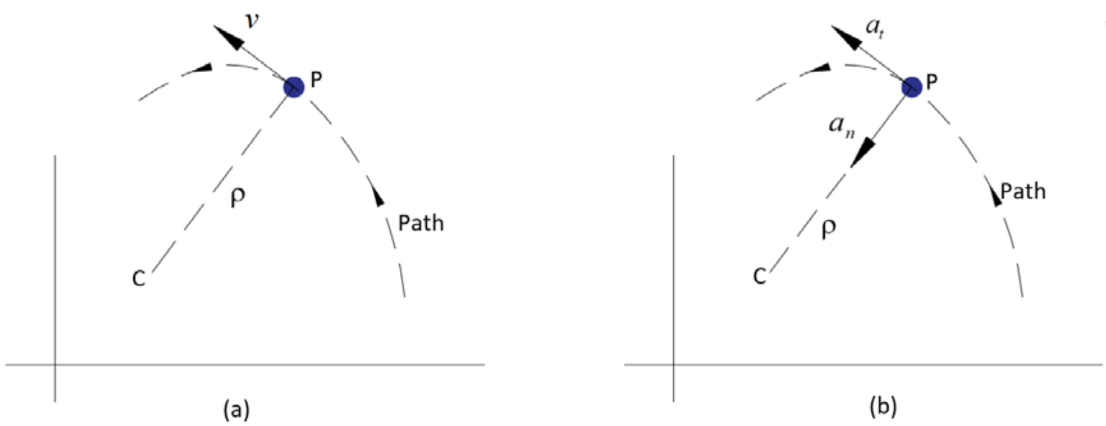


Figure 4 Normal and tangential coordinates

Acceleration will have two components, as shown in Figure 4 (b). The component along the tangential axis

Tangential Acceleration $$a_t = \frac{dv}{dt}$$ Equation 19

and the component along the normal axis, where ρ is the radius of curvature of the path.

Normal Acceleration $$a_n = \frac{v^2}{\rho}$$ Equation 20

The tangential acceleration will always be tangent to the path, but its direction will depend on how the velocity is changing with time. It will point in the direction of motion if the particle is accelerating, point in the direction opposite the motion if the particle is decelerating, or it will be zero for constant velocity. The normal acceleration will always point toward the center of rotation (point C).

Example 6

A particle is moving along a circular path at a constant speed of 70 ft/s. The radius of the circular path is 1500 feet. The particle decelerates at a constant rate to come to a complete stop in 23 seconds. What is the magnitude of total acceleration at the point when the deceleration begins?

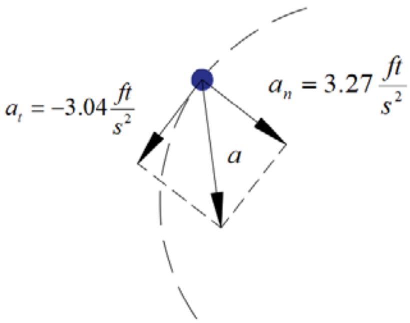
Solution:

Because the deceleration occurs at a constant rate, the tangential acceleration is determined by the average rate of deceleration.

$$a_t = \frac{dv}{dt} = \frac{v_f - v_i}{\Delta t} = \frac{(0-70)\, ft/s}{23s} = -3.04\frac{ft}{s^2}$$

The normal acceleration is based on the velocity and the radius of curvature. Because the question is asking for the total acceleration at the point when the deceleration begins, the velocity will be 70 ft/s.

$$a_n = \frac{v^2}{\rho} = \frac{\left(70\, ft/s\right)^2}{1500\, ft} = 3.27\frac{ft}{s^2}$$



The magnitude of the total acceleration is

$$a = \sqrt{a_t^2 + a_n^2}$$

$$a = \sqrt{(-3.04)^2 + 3.27^2}$$

$$a = 4.46\frac{ft}{s^2}$$

3.3.4 Polar Coordinates

The last coordinate system we will cover for curvilinear motion is the polar coordinate system. The position of the particle will be defined by its radial dimension r and its angular dimension θ as shown in Figure 5. Unit vectors $\hat{e}_r$ and $\hat{e}_\theta$ are defined as shown.

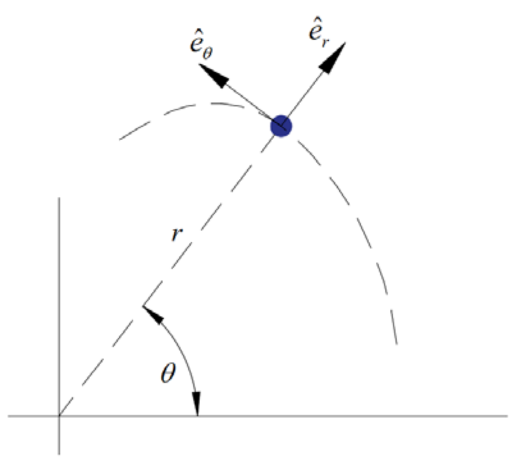


Figure 5 Polar coordinates

Without proof, the velocity vector is defined as

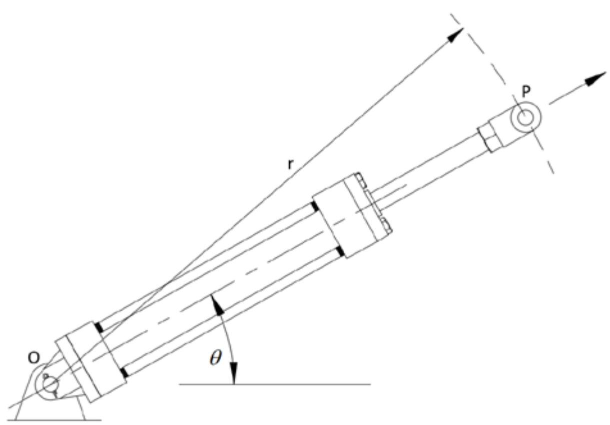
$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \qquad \text{Equation 21}$$

and the acceleration vector is defined as

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta \qquad \text{Equation 22}$$

Example 7

A hydraulic cylinder rotates about point O. Rotation of the cylinder is defined by $\theta = 0.3t$, where θ is in radians and time is in seconds. As the cylinder rotates, the rod extends to give $r = 24 + 1.3t + 0.2t^2$, where r is in inches. Determine the magnitude of the velocity and acceleration of point P at a time of 2 seconds.



Solution:

We need to determine the time derivatives of $\theta(t)$ and $r(t)$.

$$\theta = 0.3t \qquad r = 24 + 1.3t + 0.2t^2$$

$$\dot{\theta} = 0.3 \qquad \dot{r} = 1.3 + 0.4t$$

$$\ddot{\theta} = 0 \qquad \ddot{r} = 0.4$$

These equations can now be used to determine velocity

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta = (1.3 + 0.4t)\hat{e}_r + \left(24 + 1.3t + 0.2t^2\right)(0.3)\hat{e}_\theta$$

and acceleration equations for point P.

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta$$

$$\vec{a} = \left(0.4 - \left(24 + 1.3t + 0.2t^2\right)(0.3)\right)\hat{e}_r + \left(2(1.3 + 0.4t)(0.3)\right)\hat{e}_\theta$$

The velocity and acceleration expressions can now be evaluated at a time of 2 seconds.

$$\vec{v} = (1.3 + 0.4(2))\hat{e}_r + \left(24 + 1.3(2) + 0.2(2)^2\right)(0.3)\hat{e}_\theta = 2.1\hat{e}_r + 9.24\hat{e}_\theta$$

$$\vec{a} = \left(0.4 - \left(24 + 1.3(2) + 0.2(2)^2\right)(0.3)\right)\hat{e}_r + \left(2(1.3 + 0.4(2))(0.3)\right)\hat{e}_\theta = -7.82\hat{e}_r + 1.26\hat{e}_\theta$$

The magnitudes can be determined using

$$v = \sqrt{(2.1)^2 + (9.24)^2} = 9.48\frac{in}{s}$$

$$a = \sqrt{(-7.82)^2 + (1.26)^2} = 7.92\frac{in}{s^2}$$

4.0 Kinematics of Rigid Bodies

4.1 Introduction

Next, we move into the category of problems relating to kinematics of rigid bodies. Planar motion of a rigid body will fall into one of three categories: translation, pure rotation, or general plane motion.

4.2 Translation

In translational motion, any line segment on the rigid body will remain in the same orientation throughout the entire motion. Translation is further subdivided into rectilinear translation and curvilinear translation. In rectilinear motion, all points on the rigid body move along straight lines. In curvilinear translation, all points on the rigid body move along congruent curved paths. Both are illustrated in Figure 6 using a pallet moving along a monorail conveyor system. In both examples, the pallet remains parallel to the ground (does not rotate).

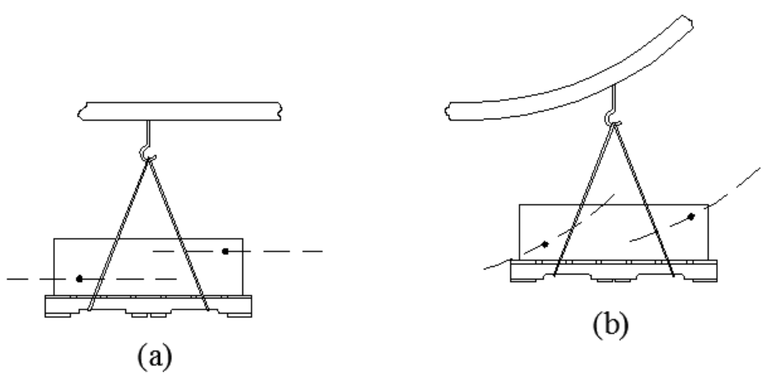


Figure 6 (a) Rectilinear translation (b) Curvilinear translation (Doane, 2015) *[reused with permission]*

Figure 6 (a) shows the pallet moving along a straight line. The trajectory of two points shown will be straight lines, which is rectilinear translation. In Figure 6 (b) the pallet is moving along a curved portion of the monorail system, but the pallet remains in a level position. The trajectory of the two points will now be curved paths while the pallet remains parallel to the floor, which is curvilinear translation.

4.3 Rotation about a Fixed Axis

The next type of motion to consider is rotation about a fixed axis, which is sometimes referred to as pure rotation. A point on a body in pure rotation will have a circular trajectory. Multiple points on a body in pure rotation will have trajectories making concentric circles centered about the fixed rotation point. Gears, as illustrated in Figure 7, are a common engineering example of pure rotation. Each gear in the system rotates on a fixed shaft.

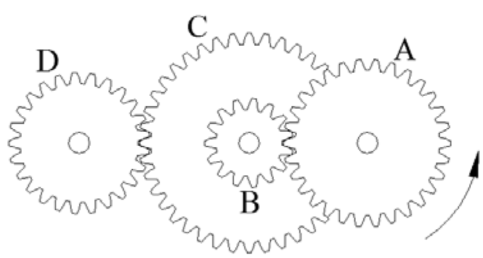


Figure 7 System of gears in pure rotation (Doane, 2015) *[reused with permission]*

4.3.1 Velocity

If the particle is moving in a curved path, the velocity is expressed as the change in angular position divided by the change in time. Instantaneous angular velocity (or rotational velocity) is time rate of change of angular displacement, as expressed in Equation 23. For angular velocity, θ is the angular position and t is time.

Angular Velocity $$\omega = \frac{d\theta}{dt}$$ Equation 23

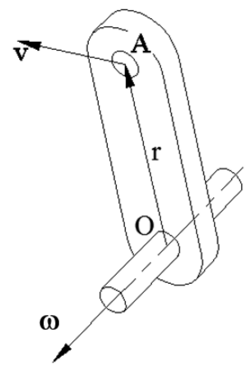


Figure 8 Velocity for fixed axis rotation (Doane, 2015) *[reused with permission]*

Tangential velocity and rotational velocity are related as shown in Figure 8. The instantaneous linear velocity of a point on a rotating body will be proportional to the distance of that point from the center of rotation. The velocity of point *A* is defined by the cross product shown in Equation 24.

$$\vec{v} = \vec{\omega} \times \vec{r}$$ Equation 24

In planar motion the velocity can be expressed in the scalar form shown in Equation 25.

$$v = r\omega$$ Equation 25

4.3.2 Acceleration

Angular acceleration is used if the motion occurs on a curved path, and the equation for angular acceleration is shown in Equation 26.

Angular Acceleration $$\alpha = \frac{d^2\theta}{dt^2} = \frac{d\theta}{dt}$$ Equation 26

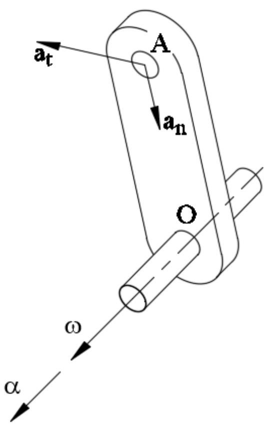


Figure 9 Acceleration for fixed axis rotation (Doane, 2015) *[reused with permission]*

Acceleration is positive if velocity is increasing and negative if velocity is decreasing (deceleration). Velocity and acceleration are both vector quantities, so they both have magnitude and direction.

The absolute accelerations for pure rotation can be determined from differentiation of the velocity terms. The direction of the acceleration vector is more complicated than that of velocity. Acceleration is typically separated into two components: tangential and normal. As illustrated in Figure 9, the rotational speed of the body is $\vec{\omega}$ and the rotational acceleration is $\vec{\alpha}$.

With the position vector $\vec{r}$ being the vector from point O to point A, the accelerations are determined from the following cross products.

$$\begin{aligned} \vec{a}_t &= \vec{\alpha} \times \vec{r} \\ \vec{a}_n &= \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) \end{aligned} \qquad \text{Equation 27}$$

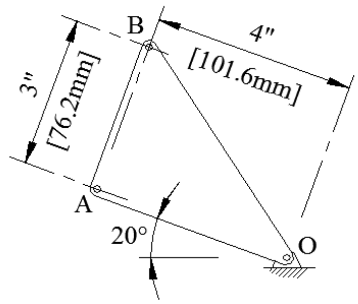
For planar motion the accelerations can also be represented in scalar form. The rotational velocity and acceleration will always be in the $\pm\hat{k}$ direction, counterclockwise rotation being positive. The magnitude of the tangential and normal accelerations will be

$$\begin{aligned} a_t &= r\alpha \\ a_n &= r\omega^2 \end{aligned} \qquad \text{Equation 28}$$

and their directions can be determined by inspection.

Example 8

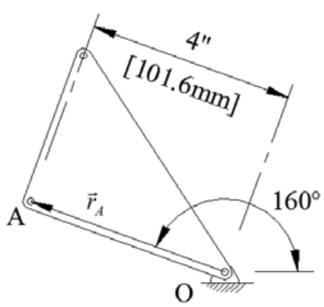
The right triangular plate shown in figure 2.7 is in pure rotation about point O and is decelerating at a rate of 3 rad/sec. At the instant shown it has a rotation speed of 5 rad/sec clockwise. Determine the velocity, tangential acceleration, and normal acceleration vectors for points A and B.



Solution:

The rotational speed and acceleration can both be expressed as vectors. The rotation speed is clockwise, which is a negative vector. The deceleration indicates the acceleration direction is counterclockwise, which is positive.

$$\vec{\omega} = -5\hat{k}\ rad/s \quad \vec{\alpha} = 3\hat{k}\ rad/s^2$$



We will start with point A. The position vector shown will be

$$\vec{r}_A = 4\cos(160)\hat{i} + 4\sin(160)\hat{j}$$

$$\vec{r}_A = -3.7588\hat{i} + 1.3681\hat{j}$$

The velocity of point A will be

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$\vec{v}_A = \vec{\omega} \times \vec{r} = -5\hat{k} \times \left(-3.7588\hat{i} + 1.3681\hat{j}\right)$$

$$\vec{v}_A = 6.8405\hat{i} + 18.7940\hat{j}$$

The acceleration of point A will be

$$\left(\vec{a}_A\right)_t = \vec{\alpha} \times \vec{r} = 3\hat{k} \times \left(-3.7588\hat{i} + 1.3681\hat{j}\right)$$

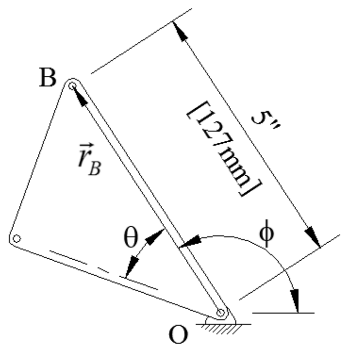
$$(\vec{a}_A)_t = -4.1\hat{i} - 11.3\hat{j} \quad [in/s^2]$$

$$(\vec{a}_A)_n = \vec{\omega} \times (\vec{\omega} \times \vec{r}) = -5\hat{k} \times \left(-5\hat{k} \times \left(-3.7588\hat{i} + 1.3681\hat{j}\right)\right)$$

$$(\vec{a}_A)_n = -5\hat{k} \times \left(6.8405\hat{i} + 18.7940\hat{j}\right)$$

$$(\vec{a}_A)_n = 94.0\hat{i} - 34.2\hat{j} \quad [in/s^2]$$

Next, we move to point B. The angles shown in the figure below need to be determined.



$$\tan\theta = \frac{3}{4} \quad \Rightarrow \quad \theta = 36.87°$$

$$\phi = 160° - \theta = 160° - 36.87° = 123.13°$$

Length OB is 5" because the triangle is a 3-4-5 triangle. Position vector for point B is

$$\vec{r}_B = 5\cos(123.13)\hat{i} + 5\sin(123.13)\hat{j}$$

$$\vec{r}_B = -2.7327\hat{i} + 4.1872\hat{j}$$

The velocity of point B will be

$$\vec{v}_B = \vec{\omega} \times \vec{r} = -5\hat{k} \times \left(-2.7327\hat{i} + 4.1872\hat{j}\right)$$

$$\vec{v}_B = 20.9360\hat{i} + 13.6635\hat{j} \quad [in/s]$$

Cross products are used to get acceleration values for point B.

$$(\vec{a}_B)_t = \vec{\alpha} \times \vec{r} = 3\hat{k} \times \left(-2.7327\hat{i} + 4.1872\hat{j}\right)$$

$$(\vec{a}_B)_t = -12.6\hat{i} - 8.2\hat{j} \quad [in/s^2]$$

$$(\vec{a}_B)_n = \vec{\omega} \times (\vec{\omega} \times \vec{r}) = -5\hat{k} \times \left(-5\hat{k} \times \left(-2.7327\hat{i} + 4.1872\hat{j}\right)\right)$$

$$(\vec{a}_B)_n = -5\hat{k} \times \left(20.9360\hat{i} + 13.6635\hat{j}\right)$$

$$\left(\vec{a}_B\right)_n = 68.3\hat{i} - 104.7\hat{j} \quad [in/s^2]$$

4.4 General Plane Motion

Planar motion that is a combination of translation and pure rotation is known as general plane motion. Consider the slider-crank mechanism shown in Figure 10. The coupler (link 3) will move in general plane motion. The general plane motion can be broken down into translation plus rotation, as shown in Figure 10 (b).

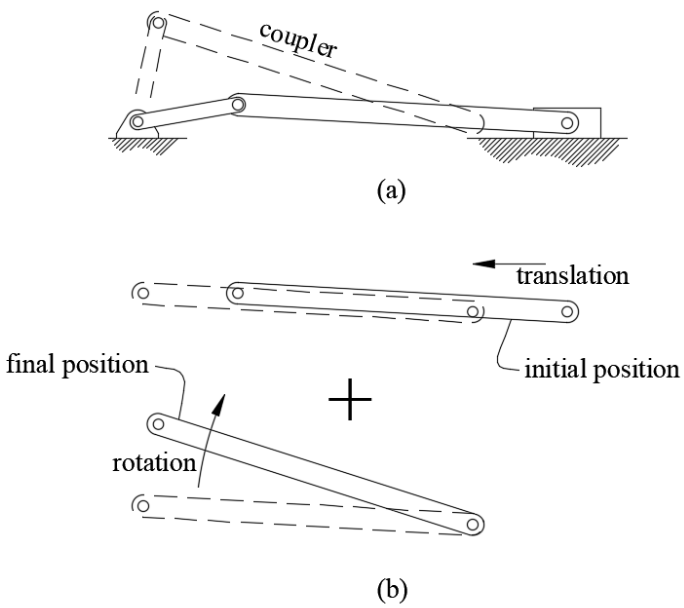


Figure 10 General plane motion (Doane, 2015) *[reused with permission]*

4.4.1 Velocity

4.4.1.1 Velocity Difference and Relative Velocity

Velocity analysis becomes more complex with general plane motion because the rotation does not occur about a fixed pivot point. Figure 11 (a) shows a link pivoting about point B. Unlike pure rotation, point B is now moving on a slide. Motion described relative to another moving point is called relative motion.

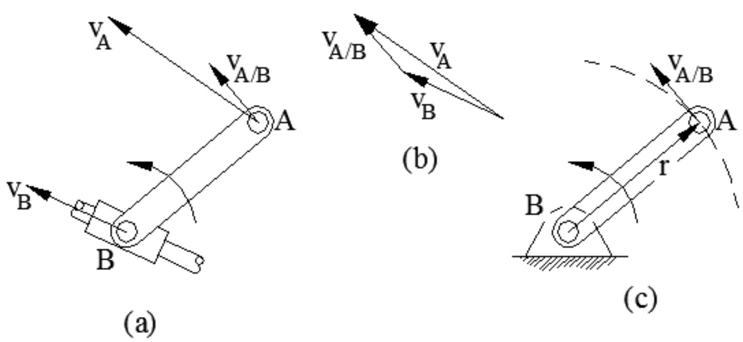


Figure 11 (a) Velocity difference (b) Vector polygon (c) Representation of the velocity of point *A* with respect to point *B* (Doane, 2015) *[reused with permission]*

As represented by the vector polygon in Figure 11 (b), the absolute velocity of point *A* ($\vec{v}_A$) now is determined using velocity difference.

$$\vec{v}_A = \vec{v}_B + \vec{v}_{A/B} \qquad \text{Equation 29}$$

As shown in Figure 11 (c), the velocity of point *A* with respect to point *B* $(\vec{v}_{A/B})$ is determined by treating point *B* as a fixed pivot and determining the velocity of point *A* as if it were in pure rotation.

$$\vec{v}_{A/B} = \vec{\omega}_{AB} \times \vec{r}_{A/B} \qquad \text{Equation 30}$$

Substituting Equation 30 into Equation 29 gives

$$\vec{v}_A = \vec{v}_B + \vec{\omega}_{AB} \times \vec{r}_{A/B} \qquad \text{Equation 31}$$

The vector equation can be split into two scalar equations, which can be solved for the two unknowns $\vec{v}_A$ and $\vec{\omega}_{AB}$. An example of using velocity difference will be presented in the application problem (Example 10) in Section 4.5.

4.4.1.2 Instant Center of Rotation

A rigid body in plane motion can always be thought of as being in pure rotation about some fixed instantaneous center of rotation. The location of the instant center will change for a body in general plane motion, and note that its position will not necessarily be located on the actual rigid body. The velocity of the instant center will be zero; however, the acceleration will generally not be zero. Therefore, the instant center of rotation is only used in velocity analysis.

The instant center of rotation can be found on any rigid body in general plane motion if the velocity directions of two points are known. Consider the rigid body in Figure 12 (a) where the direction of the velocities at points A and B are known. The velocity vector is always perpendicular to the radial line to the center of rotation. Therefore, if a line is drawn perpendicular to each known velocity vector, the intersection of those two lines will be the instant center of rotation (labeled IC on the figure). The direction of velocity vectors at any other point on the rigid body can then be determined. Figure 12 (b) shows lines drawn from the instant center to points C and D. The velocity at each point must be perpendicular to those lines. The directions of the velocity must be consistent with the rotational direction of the rigid body and the magnitudes can be determined based on the distance of that point to the instant center.

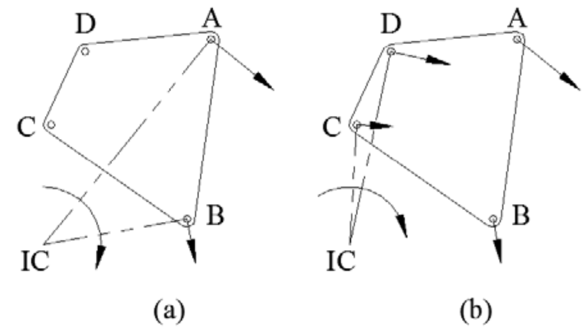
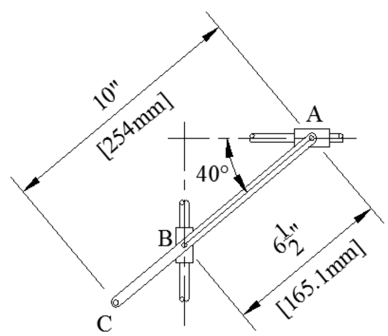


Figure 12 (a) Locating the instant center of rotation (b) Determining velocity of points using the instant center (Doane, 2015) *[reused with permission]*

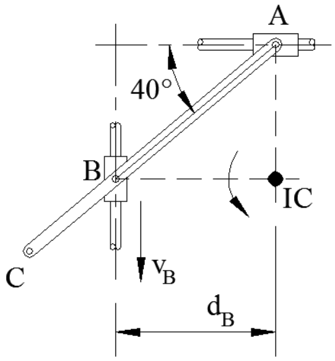
Example 9

Link ABC shown is connected to two blocks. At the instant shown block B has a downward velocity of 1.2 m/s. Determine the angular velocity of the rod.



Solution:

The velocity of point B is downward and the velocity of point A will be constrained to a horizontal direction. The intersection of lines drawn perpendicular to the velocities will be the instant center.



The distance from B to the instant center is required to determine the angular velocity of the rod.

$$\cos 40 = \frac{d_B}{0.1651m} \quad \Rightarrow \quad d_B = 0.126m$$

Because the rod can be considered in pure rotation about the instant center the angular velocity of the bar can now be determined from the known velocity and radial distance from the instant center.

$$\omega = \frac{v_B}{d_B} = \frac{1.2\frac{m}{s}}{0.126m} = 9.5\frac{rad}{s} \quad [ccw]$$

4.4.2 Acceleration

Acceleration of a point relative to another moving point will be relative acceleration.

$$\vec{a}_A = \vec{a}_B + \vec{a}_{A/B} \qquad \text{Equation 32}$$

Unlike velocity analysis, the relative acceleration term will have two components. The term $\vec{a}_{A/B}$ will be rotational so it will have normal and tangential components, which were given earlier for rotation about a fixed point. Substituting the normal and tangential acceleration terms give

$$\vec{a}_A = \vec{a}_B + \vec{a}_t + \vec{a}_n$$

$$\vec{a}_A = \vec{a}_B + \vec{\alpha}_{AB} \times \vec{r}_{A/B} + \vec{\omega}_{AB} \times \left(\vec{\omega}_{AB} \times \vec{r}_{A/B}\right) \qquad \text{Equation 33}$$

An example of calculating acceleration will be presented in the application problem (Example 10) in Section 4.5.

4.5 Applications for Rigid Body Kinematics

There are many applications of rigid body kinematics in engineering. One common application in mechanical engineering would be machine dynamics. An example will be provided for a slider-crank mechanism. The mechanism is a nice example because it illustrates all types of rigid body motion. Figure 13 shows a slider-crank mechanism, which will be used in the Example 10. The crank (link OA) is in pure rotation about the fixed-point O. The piston is constrained to only move in the vertical direction, so it is in pure linear translation. The connecting rod (link AB) is in general plane motion.

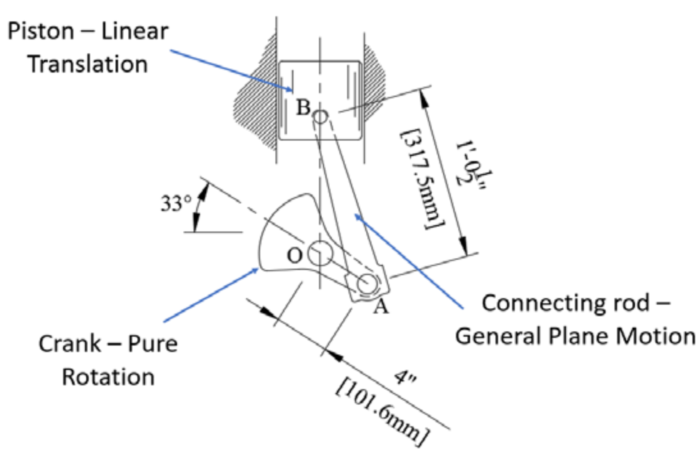
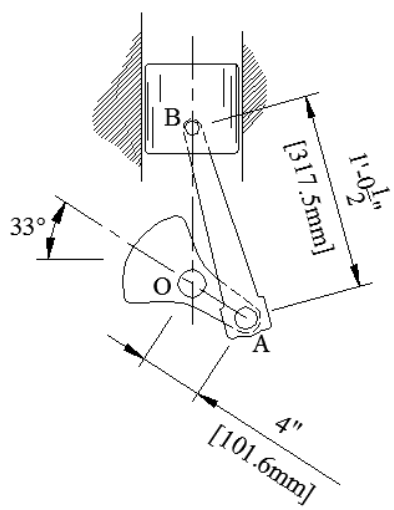


Figure 13 Component motions in a slider-crank mechanism

Example 10

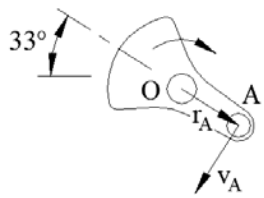
The figure shows a slider-crank mechanism. The crank (link OA) rotates at a constant speed of 1200 rpm clockwise. For the position shown determine the velocity and acceleration of the piston (point B).



Solution:

We will start with the crank (link OA), which is in pure rotation. The rotation speed needs to be converted to radians per second.

$$\omega_2 = 1200\frac{rev}{\min}\left(\frac{2\pi\, rad}{rev}\right)\left(\frac{1\min}{60\,s}\right) = 125.7\frac{rad}{s}$$

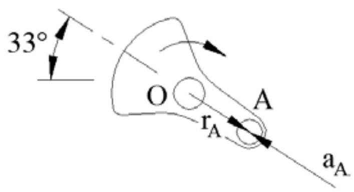


The velocity of point A can be determined.

$$\vec{v}_A = \vec{\omega}_2 \times \vec{r}_A = -125.7\hat{k} \times \left(4\cos(-33)\hat{i} + 4\sin(-33)\hat{j}\right)$$

$$\vec{v}_A = -273.85\hat{i} - 421.69\hat{j} \quad [in/s]$$

The tangential acceleration of point A will be zero because the rotational acceleration is zero (rotates at a constant speed). The normal component will not be zero.



$$\vec{a}_A = \vec{\omega}_2 \times \left(\vec{\omega}_2 \times \vec{r}_A\right)$$

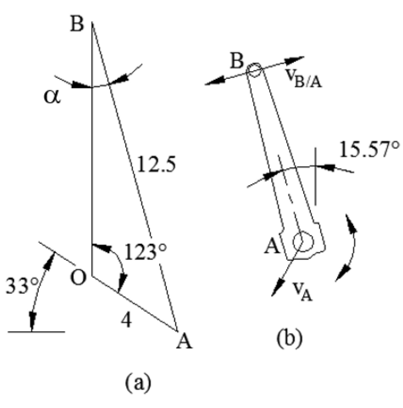
$$\vec{a}_A = -125.7\hat{k} \times \left(-125.7\hat{k} \times \left(4\cos(-33)\hat{i} + 4\sin(-33)\hat{j}\right)\right)$$

$$\vec{a}_A = -125.7\hat{k} \times \left(-125.7\hat{k} \times \left(3.3547\hat{i} + -2.1786\hat{j}\right)\right)$$

$$\vec{a}_A = -125.7\hat{k} \times \left(-273.85\hat{i} - 421.6858\hat{j}\right)$$

$$\vec{a}_A = -53006\hat{i} + 34423\hat{j} \quad [in/s^2]$$

Next, we move on to the connecting rod (link AB). The angular position of the connecting rod can be determined using law of sines.



$$\frac{12.5}{\sin 123} = \frac{4}{\sin\alpha} \quad \Rightarrow \quad \alpha = 15.57°$$

The velocity of point B is determined using the relative velocity equation.

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} = \vec{v}_A + \vec{\omega}_3 \times \vec{r}_{B/A}$$

As shown, the direction of the rotational velocity of link 3 is currently unknown. Therefore, the direction of the relative velocity vector is unknown (though it lies on a line perpendicular to line AB).

Assuming a counterclockwise rotational velocity gives

$$\vec{v}_B = -273.85\hat{i} - 421.69\hat{j} + \omega_3\hat{k} \times \left(-12.5\sin(15.57)\hat{i} + 12.5\cos(15.57)\hat{j}\right)$$

$$\vec{v}_B = -273.85\hat{i} - 421.69\hat{j} - 12.04\omega_3\hat{i} - 3.36\omega_3\hat{j}$$

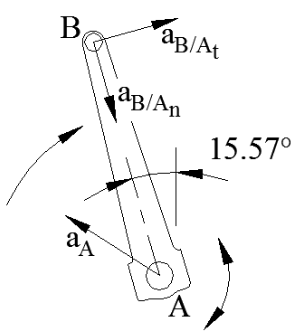
The vector equation can be separated into two scalar equations. Taking the x-direction equation (note that the velocity of B is vertical so its x-component will be zero)

$$0 = -273.85 - 12.04\omega_3 \quad \Rightarrow \quad \omega_3 = -22.74\ rad/s$$

The negative sign indicates that the rotation direction is clockwise. Taking the y-direction equation

$$v_B = -421.69 - 3.36(-22.74)$$

$$v_B = 345.2\ in/s \downarrow$$



The acceleration of point B is determined using relative acceleration.

$$\vec{a}_B = \vec{a}_A + \vec{\alpha}_3 \times \vec{r}_{B/A} + \vec{\omega}_3 \times (\vec{\omega}_3 \times \vec{r}_{B/A})$$

The direction of the rotational acceleration is again unknown, so we will assume a positive counterclockwise direction.

$$\vec{a}_B = -53006\,\hat{i} + 34423\,\hat{j} + \alpha_3\hat{k} \times (-12.5\sin(15.57)\hat{i} + 12.5\cos(15.57)\hat{j})$$
$$- 22.74\hat{k} \times (-22.74\hat{k} \times (-12.5\sin(15.57)\hat{i} + 12.5\cos(15.57)\hat{j}))$$

$$\vec{a}_B = -53006\,\hat{i} + 34423\,\hat{j} + \alpha_3\hat{k} \times (-3.36\hat{i} + 12.04\hat{j})$$
$$- 22.74\hat{k} \times (-22.74\hat{k} \times (-3.36\hat{i} + 12.04\hat{j}))$$

$$\vec{a}_B = -53006\,\hat{i} + 34423\hat{j} - 12.04\alpha_3\hat{i} - 3.36\alpha_3\hat{j} + 1737.47\hat{i} - 6225.98\hat{j}$$

$$\vec{a}_B = -51268\,\hat{i} + 28197\,\hat{j} - 12.04\alpha_3\hat{i} - 3.36\alpha_3\hat{j}$$

The vector equation can be separated into two scalar equations. We will again start with the x-direction equation because acceleration of point B will be purely vertical.

$$0 = -51268 - 12.04\alpha_3 \quad \Rightarrow \quad \alpha_3 = -4134\,[rad/s^2]$$

The y-direction equation gives

$$a_B = 28197 - 3.36(-4134)$$

$$a_B = 3507 \;\; ft/s^2 \uparrow$$

5.0 Brief Review of Calculus

This section is not intended to provide a full review of calculus. However, it will provide a very brief review of the calculus needed for this course. Simple polynomial function will be used in this course because the derivative and integrals for such functions are very basic. More complex calculus, such as chain rule of differentiation and integration by parts, will not be discussed here or used in this course.

5.1 Some Rules of Differentiation

Kinematics requires differentiation to find velocity and acceleration. This section will provide the basic rules of differentiation required for this course.

Derivatives give the slope of a curve at a specific point. One thing to keep in mind is that derivatives can be written in different ways. For example, the derivative of a function $f(x)$ with respect to x can be written as $\frac{d}{dx}f(x)$ or $f'(x)$. Time derivatives (derivatives of a function with respect to time) are sometimes expressed in dot notation. So, $\frac{d}{dt}x$ can be expressed as $\dot{x}$.

5.1.1 The Constant Rule

The derivative of a constant function is zero. If c is a constant, the derivative is expressed as

$$\frac{d}{dx}[c] = 0 \qquad \text{Equation 34}$$

5.1.2 The Power Rule

If the variable x is raised to a power n, where n is any real number, the derivative with respect to x is defined by

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1} \qquad \text{Equation 35}$$

So, as a general example, the derivative of the function $f(x) = x^3$ with respect to x would be

$$\frac{d}{dx}\left[x^3\right] = 3x^2$$

Note that the rule applies to negative values of n. For example, the derivative of the function $f(x) = \frac{1}{x^2}$ can be determined by rewriting the function as $f(x) = x^{-2}$ and apply the rule given in Equation 35.

$$\frac{d}{dx}\left[x^{-2}\right] = -2x^{-3} = \frac{-2}{x^3}$$

For the case when n = 1, the derivative is

$$\frac{d}{dx}[x] = 1 \qquad \text{Equation 36}$$

5.1.3 Constant Multiple Rule

Constants can be "pulled out" of the derivative. If c is a real number

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] \qquad \text{Equation 37}$$

As an example

$$\frac{d}{dt}\left[\frac{4}{5}t^2\right] = \frac{4}{5}\cdot\frac{d}{dt}\left[t^2\right] = \frac{4}{5}(2t) = \frac{8}{5}t$$

5.1.4 Sum and Difference Rule

The derivative of more complex polynomial functions can be solved using the sum and difference rule.

$$\begin{aligned} \frac{d}{dx}[f(x)+g(x)] &= f'(x)+g'(x) \\ \frac{d}{dx}[f(x)-g(x)] &= f'(x)-g'(x) \end{aligned} \qquad \text{Equation 38}$$

As an example,

$$\frac{d}{dx}\left[2x^3+3x\right]=6x^2+3$$

5.2 Some Rules of Integration

Integration gives area under a curve. An integral is an antiderivative.

$$\int f(x)dx = F(x)+C$$

Where $F'(x)=f(x)$ and C is a constant

Equation 39

5.2.1 The Constant Rule

If k is a constant, the integral of the constant is defined by

$$\int kdx = kx + C$$

Equation 40

5.2.2 The Power Rule

For a case where x is raised to a power n, where n is a real number not equal to -1, the integration is defined as

$$\int x^n dx = \frac{x^{n+1}}{n+1}+C$$

Equation 41

5.2.3 Constant Multiple Rule

If k is a constant multiplied by a function, the constant can be "pulled out" of the integration.

$$\int kf(x)dx = k\int f(x)dx \qquad \text{Equation 42}$$

5.2.4 Sum and Difference Rule

Integration of more complex polynomial functions can be solved using the sum and difference rule.

$$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$
$$\int [f(x)-g(x)]dx = \int f(x)dx - \int g(x)dx \qquad \text{Equation 43}$$

6.0 Works Cited

Doane, J. (2015). *Machine Analysis with Computer Applications.* Wiley.

