A SunCam online continuing education course

Structural Nonlinearity: Analysis Methods

by

John Klein, P.E., M.L.S.E.

Table of Contents

Introduction

Started in 1965, NASTRAN (**NA**SA **STR**ucture **AN**alysis) was one of the first software packages ever developed yet nonlinear analysis methods continue to emerge and evolve today. There are graduate level university courses solely focusing on nonlinear analysis methods and this course does not attempt to substitute for that level of knowledge.

This course provides an overview of mainstream analysis methods and shows the applicability of analysis methods to different nonlinearity types. One detailed example is included to demonstrate how iterative nonlinear analyses determine solutions. The scope of this course is limited to statics and generally presents concepts and examples as planar to leave the complexity to the topic at-hand.

Review of Structural Nonlinearity

Structural nonlinearity can be defined as a structural system that results in having stiffness analysis components that are not constant.

A structural system is nonlinear if an accurate stiffness matrix or load vector for use in a linear stiffness analysis would contain expressions instead of numerical values, and the expressions include variable(s) such as:

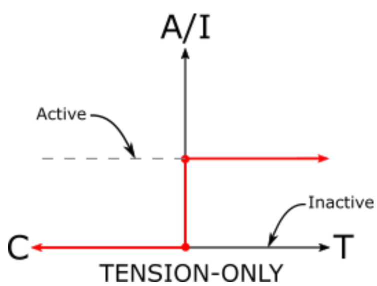
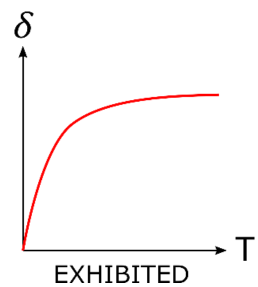
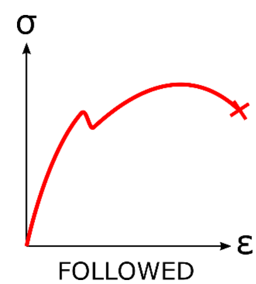
- member/joint deflection, slope, displacement, or rotation
- the location along the length of a member
- direction of force(s)
- extent of strain
- magnitude of reaction
- time

<u>A structural system with one or more instances of acting/engaged structural nonlinearity cannot be analyzed by only a single linear stiffness analysis.</u>

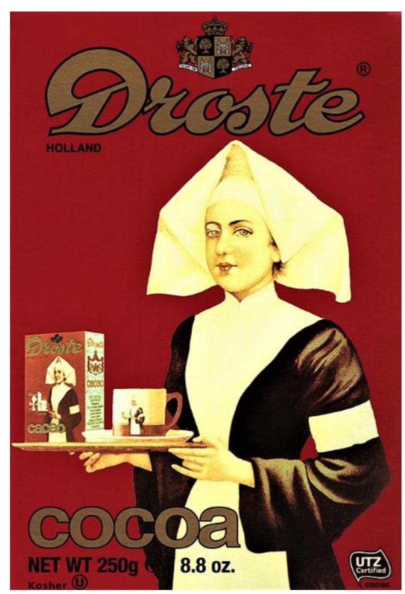
The sources of structural nonlinearity can be summarized as:

<u>Geometric nonlinearity</u>	<u>Contact</u>	<u>Time</u>
<u>Material nonlinearity</u>	<u>Shape</u>	

Refer to the first course in this series for a comprehensive list of the different possible types of structural nonlinearities and for general guidelines on when to include the different types. Remember that phase-type nonlinearities have binary, on/off states, while continuum-type nonlinearities either exhibit nonlinear behavior that can be graphed as a curve or nonlinearly behave by following a given curve.

Phase-type nonlinearities such as gap supports and directional members	Continuum-type nonlinearities P-δ behavior (left) & material nonlinearity (right)
	 

All nonlinearities that are possible in a structural system should be considered. Consider whether the nonlinearity is present, if it will help emulate real-world behavior, and if it will be non-negligible, influential, and consequential. If consideration leads to including nonlinearity in analyses then the analysis method chosen will need to be able to solve that type of nonlinearity and any other nonlinearities that are concurrently in the same structural system. Concurrent nonlinearities must be analyzed concurrently as nonlinear results cannot be combined or superimposed, and nonlinear results cannot be extrapolated or scaled (up).



The nature of Nonlinear Analyses

At the heart of nonlinear analysis is the circular reference. Nonlinear analysis results are a function of values that are also results of the analysis. This is called *recursion*. Recursion is when something is defined in terms of itself. In the art world recursion is cleverly embraced in pieces that exhibit the Droste effect. The Droste effect shows an image, which contains a smaller copy of that same image, which contains another smaller copy, and so on.

How does an artist resolve the infinite? By replicating the image over and over until it is so small that it no longer matters, then they stop. This is very similar to how nonlinear structural analysis works.

Analysis ingredients

Analyzing a structural system requires a user, a method, and a tool(s). The user needs to know the requirements, parameters, and constraints of the analysis, and the user needs to be able to somewhat anticipate structural behavior and understand the method to be implemented. As possible the user should have experience with the method, the tool, and maybe even the specific method-tool combination.

<u>Methods</u>

The method is the special part. Users can gain experience and learn, tools can be quickly built, but methods must be discovered, developed, tested, refined, and proven. When multiple methods are recognized and available then method choice means weighing many factors. Most importantly the method needs to be capable and applicable.

When the methods have explicitly descriptive names then the decisions can be obvious. Few would hesitate when deciding between the “highly precise method” and the “approximate method” for testing to locate tendons before drilling through an existing, elevated, post-tensioned concrete slab. If cost, schedule, and personnel training is no issue then the decision is again easy, just use the most sophisticated, capable method. Otherwise, <u>the choice of analysis method for solving structural nonlinearity should focus first on capability and applicability</u>, then familiarity, efficiency, usability, accessibility, affordability, and other factors as best suits the situation.

<u>Analysis Tools</u>

There are tools available if structural nonlinearities need to be analyzed. The tools can be roughly categorized as either *rudimentary* or *aided*. Rudimentary analyses would include hand calculations, tabulated data, approximate methods, or manually scripting/coding using spreadsheets, Mathcad, MATLAB, R, Python, etc. Aided analyses are performed using packaged software.

<u>Idealization</u>

It is rare for engineers to encounter structural systems where some interpretation of the real-world is not required to structurally analyze the system. Guidance on idealizing structures for nonlinear analysis is presented in the next course that focuses on analyzing nonlinearity.

> Structural movement includes displacements, rotations, deflections, and slopes of structures and their members. The term *distort* will be used as appropriate in this document to substitute for structural movement.
>
> The term *P-delta* is used in this document to collectively refer to both P-δ and P-Δ effects of geometric nonlinearity.

Analysis Methods

Linear / Stiffness / 1st-Order analysis

The classic stiffness analysis is a powerful tool and a workhorse of the structural engineering industry. A stiffness matrix and load vector are formulated for the structure and then solved to a find the displacement vector. This type of analysis is also called a displacement analysis, the direct stiffness method, or the matrix stiffness method.

$$\{F\} = [K]\{u\}$$

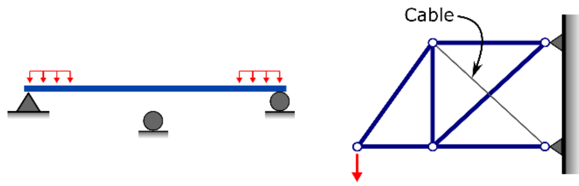
$$\begin{bmatrix} F_{x1} \\ \vdots \\ F_{yn} \end{bmatrix} = \begin{bmatrix} K_{11} & \cdots & K_{1n} \\ \vdots & \ddots & \vdots \\ K_{n1} & \cdots & K_{nn} \end{bmatrix} \begin{bmatrix} u_{x1} \\ \vdots \\ u_{yn} \end{bmatrix}$$

This type of analysis is what typical *structural analysis software* (SAS) uses. It is also at the core of many of the analysis methods used to analyze nonlinearity, though a single linear analysis (*1st-order analysis*) cannot typically produce nonlinear results. This is not to say that linear analyses are not useful. The following are ways that a 1st-order analysis can be useful in the context of nonlinearity:

- as a validity check of a structural model before adding/including nonlinearity, and as a tool to obtain some schematic-level analysis insight
- as a comparison against later nonlinear model results
- as the initial analysis step of a repeated subroutine in an iterative nonlinear analysis (much more on this below)

Note that attempting to analyze nonlinearity only by simplifying to a single linear state will yield wholly erroneous results in most cases. The only ways that a single linear analysis could substitute for a nonlinear analysis include:

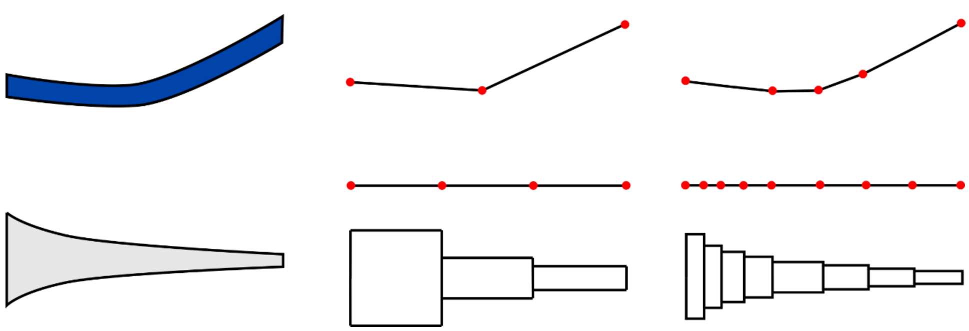
- linearly analyzing a structure that contains phase-type nonlinearities that do not get triggered/engaged for the applied loads



- using an approximation for member elastic curve formulae to analyze some geometrically nonlinear effects
- using discretization as an approximation for analyzing physically nonlinear members (nonlinearity by shape)

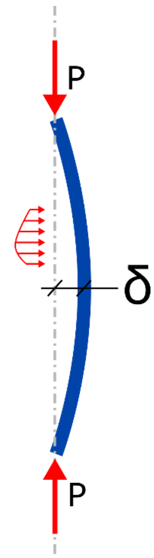
<u>Discretization</u>

Discretizing members that change along their length and analyzing the resulting structure is an approximate method. Discretization is performed using the essence of the finite element method by dividing up members that are nonlinear by shape into multiple sub-members with stepped properties. As with finite element meshes this density of sub-members is a continuum. Discretization can be a valid and quite accurate analysis method when using enough sub-members and becomes more inaccurate with fewer members. The rate of change of shape or curve should also be considered when determining discretization density. The following shows a curved member and a tapered member discretized roughly and then more finely in areas of higher gradient. The tapered sub-member depths are pictorially shown as stepped.



Another application of discretization in nonlinear analysis concerns geometric nonlinearity. Most software only considers the distorted geometry at joints or nodes when evaluating geometric nonlinearity and more nodal results help the solver converge accurately. This means that is almost always advantageous to discretize members subjected to geometric nonlinearity that involves transverse member deflections. Discretizing members that are axially loaded and subject to geometrically nonlinear P-δ effects is commonly recommended by SAS manuals, and is discussed further in the next two courses.

Geometric Stiffness Matrix



Geometrically nonlinear P-δ effects will result in members exhibiting higher or lower transverse stiffness under axial tension or compression, respectively. This is often referred to as *stress stiffening* or *stress softening* effects. The typical P-δ effect with axial compression and transverse member loads manifests as a softening effect. This softening effect was described in the first course in this series as a *mechanism*. As a column deflects laterally a moment due to axial compression eccentricity develops and tends to further deflect the column, even without any additional loads.

There are many other effects and behaviors related to geometric nonlinearity but this axial plus flexure stiffening/softening effect can be approximated to a reasonable accuracy using the *geometric stiffness matrix*. By assuming that the deflected shape of the member can be modeled using a cubic equation and deriving strain expressions for a slightly deflected member (increment), the principle of virtual displacements can be used to derive the geometric stiffness matrix[1]. This symmetric matrix allows for the flexural stiffness of members to be reduced or increased based on the magnitude of axial compression or tension, given known displacement and rotation values.

$$[k_g] = \frac{P}{L}\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ & \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ & & \frac{2L^2}{15} & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ & & & 1 & 0 & 0 \\ & & & & \frac{6}{5} & -\frac{L}{10} \\ & & & & & \frac{2L^2}{15} \end{bmatrix}$$

This matrix cannot be included in the linear stiffness analysis. The matrix is indefinite since the displacements and rotations must be known, so the linear stiffness analysis is first performed, then the linear stiffness matrix and the geometric stiffness matrix are added. The total stiffness of linear matrix + geometric matrix is then used to produce results that include the effects of stress softening and stiffening, usually to within 5% accuracy.

Note that the geometric stiffness matrix can only solve stress stiffening and softening effects. There are many other geometrically nonlinear effects that cannot be solved by using a geometric stiffness matrix alone, such as axial member shortening/elongation, large deflections, snap-through, membrane-action, apparent shortening, etc.

A few linear analyses

A 1st-order analysis cannot provide nonlinear results aside from using discretization for members that are nonlinear by shape. However, repeating linear analyses with appropriate adjustments being made to the problem inputs between iterations can provide viable and accurate nonlinear results (example provided below). Just two or three successive linear analyses can solve structures with a few instances of phase-type nonlinearities.

- A few iterations of linear analyses can be used to analyze directional supports and directional members. Presumptive-backcheck type calculations are used where members or supports are presumed to act or not act, and then backchecked after analyzing using that presumption. Iterate until all active/inactive state presumptions are proved correct.
- A few iterations of linear analyses and superposition can be used to analyze gap supports and some initial load/stress states, where the stresses or distribution of forces from the results for the different phases can be algebraically combined.

Kinematics

The notion of time must be acknowledged to visualize and evaluate some nonlinearity, such as geometric nonlinearity. The scope of this course is limited to statics, but *kinematics* can be employed as a useful compromise between statics and dynamics to analyze nonlinearity.

Kinematics is the evaluation of possible structural movement without regarding actual motion

Kinematics allows acknowledging and accommodating structural activity beyond the linear analysis equilibrium state; specifically, loads on the distorted structure can be analyzed.

Time is treated as a pseudo-variable in kinematics and is acknowledged wherein equilibrium states that occur after an initial point in time are acknowledged. How the structure and its forces and masses moved and arrived at those later states is ignored. Each state is a snapshot of an equilibrium state ahead in time, like stop motion animation or skipping a few pages at a time through a flip book, and at each step the stress-strain relationship, compatibility, and equilibrium must be satisfied.

Image A: "Horse gallop"

2nd-order analysis

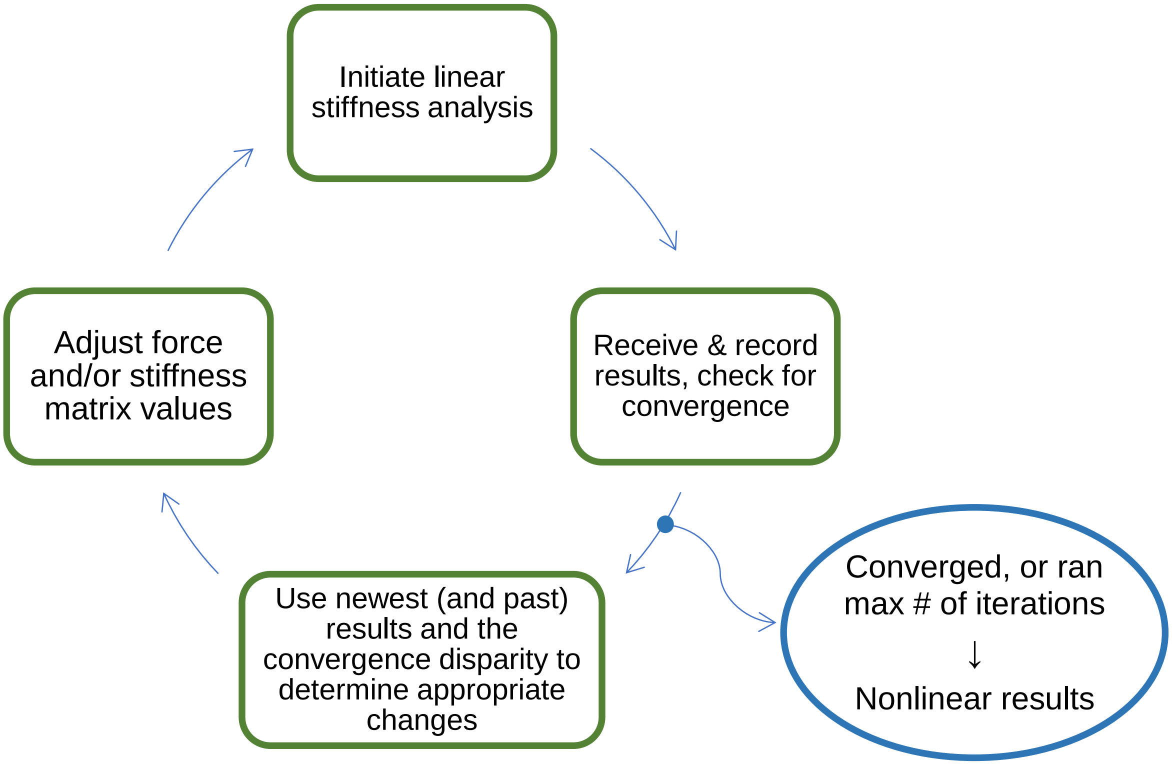
A *2nd-order analysis* is where many linear analyses are sequentially performed to solve a structure by converging to a tolerance. The analysis is orchestrated so that incrementally smaller and smaller changes are made to the inputs in between linear analysis iterations, eventually so small of change that convergence occurs with accompanying results. 2nd-order analyses are used to analyze nonlinearities that are of a more continuous nature, or when there are many instances of simple, phase-type nonlinearities. This is the usually the most advanced level of (static) nonlinear capability that conventional SAS offers and first introduces kinematics to the analysis arsenal.

A 2nd-order analysis is a *numerical analysis,* where the linear stiffness analysis is merely the computation instrument used in each step of the numerical analysis. Numerical analyses are technically approximate since they recursively converge but if the convergence threshold is high then the precision can approach exact.

> "Numerical analysis is the study of algorithms for the problems of continuous mathematics." – Dr. Lloyd Trefethen[3], 1992

> "[Numerical analysis is] using lots of simple calculations... when direct perfect answers are hard." – Mathisfun.com, 2018

The method described in the previous section ("A few linear analyses") is technically a 2nd-order analysis but was presented separately since it can be *rudimentary* if the few linear analyses, and the adjustments made between each are manually performed by the engineer. A true 2nd-order analysis will be limited to software, or perhaps coded/scripted solutions for very simple structures. The algorithm will need to:



A 2nd-order analysis with a robust algorithm, usually using the Newton-Raphson or similar method, can incrementally step through the analyses and follow moderately nonlinear behavior. With these capabilities the following nonlinearities can be analyzed:

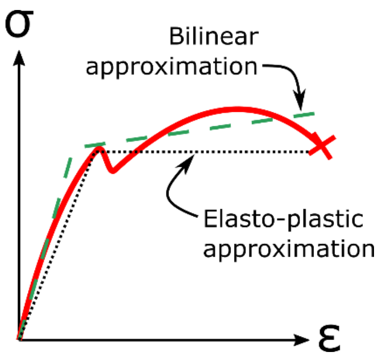
- (directional supports and members, gap-type supports, friction, initial load/stress states)
- Nonuniform supports
- Some types and extents of geometric nonlinearity, including P-δ, P-Δ, axial deformations types, and transversely loaded cables

If two advanced SAS packages were tasked with solving the same, complex, nonlinear structural system then they would be expected to converge upon slightly different results. The disparity may be on the order of tenths of a percent, but it may be surprising to then learn that the structural engineering industry generally regards both those sets of results as correct. If a robust 2nd-order solver has been successfully tested and compared to known results then it is generally regarded as an exact solution within the structural engineering industry, or at the very least *as correct as is practical*.

Many SAS packages that can perform 2nd-order analyses can also model using 2D-area and 3D-volume finite elements, though not necessarily as they are not related.

Some SAS packages that can perform 2nd-order analyses include cable elements though only a few use nonlinear cable elements. Analyzing cables and cable networks is another feature that must be researched for each specific software package.

Materially Nonlinear 2nd-order analysis



Materially nonlinear analyses are essentially the same as 2nd-order analyses but the algorithm must be capable of also monitoring all parts of the structure's progress along the material stress-strain curve and adjusting the modulus of elasticity between iterations accordingly. This method of moving along the stress-strain curve is like numerical integration and typically uses the Newton-Raphson method (or bisection method, or similar) for determining appropriate modulus adjustments. Most software packages claiming materially nonlinear capability will accommodate multilinear or curved stress-strain relationships.

A structural analysis that includes material nonlinearity of the type where a material is stressed beyond its yield point is known as *inelastic analysis* or *plastic analysis.* Which is correct and what is the difference? The only tangible difference is that plastic behavior requires a ductile material while the fracture of a brittle material could be described as inelastic. From that perspective it follows that plastic behavior is just one form of inelastic behavior, where other inelastic behaviors include materials without a linear stress-strain region (nonlinear elasticity) and fracture of brittle materials. The discussions here focus on post-yielding analysis of ductile materials.

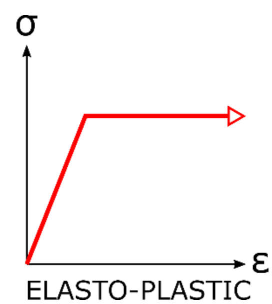
When regarding analyses that include material nonlinearity (MN) it is important to recognize and distinguish between two different MN analyses types and to understand the implications of using 1D-line elements to analyze MN. The two different types of analyses are those that acknowledge post-yielding MN and analyses that anticipate materially nonlinear behavior.

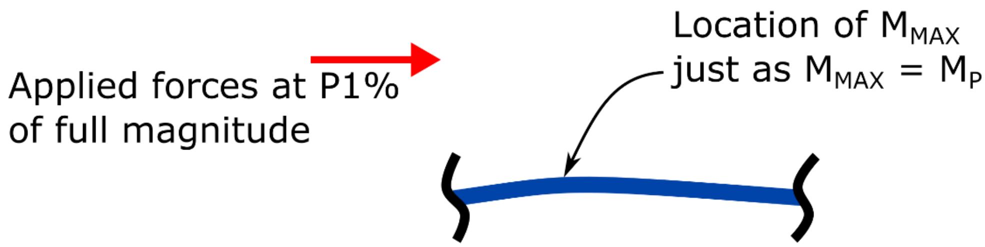
Structural analyses using only 1D-line elements like beam and truss elements do not typically include local effects. If the user understands structural mechanics and this notion they can sometimes accommodate local effects after-the-fact. Local buckling of flanges and webs can be generically checked using the calculated internal forces in those parts of members, but analyzing local effects from irregular loads or nonstandard connections, torsional warping, etc. cannot be properly analyzed using 1D-line elements. 2D-area or 3D-volume elements are required if local effects need to be analyzed.

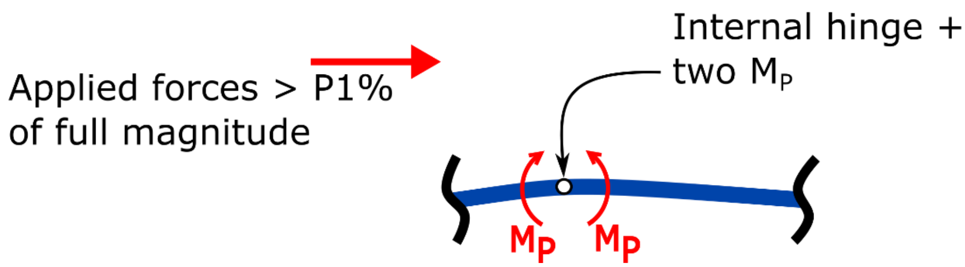
Image B: Local buckling

Local effects and instabilities are an integral part of the materially nonlinear response of most structures unless all member cross-sections are geometric solids (solid rectangles, circles, etc) and all joints and connections are oversized. This means that any analysis that uses 1D-line elements cannot provide much insight into post-yielding behavior. What do some SAS packages that use 1D-line elements and claim to provide materially nonlinear analyses actually do?

Analyses that only acknowledge material nonlinearity use yielding to an end by adopting an elasto-plastic stress-strain relationship and essentially stopping analysis when a member reaches a plastic state. By adopting a perfectly plastic post-yield response the following post-yielding simplification can be exploited.



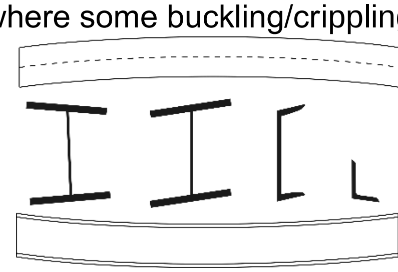




Using the simplification above can allow SAS with 2nd-order analysis capacities to use 1D-line elements and perform a simplistic *pushover analysis*. The example above used flexural plastic hinging though this simplification extends to axial forces, torsion, etc.

The alternative to using a nonlinear analysis that only acknowledges MN is to attempt to anticipate materially nonlinear inelastic behavior. The word anticipate is used purposefully here as using structural analyses to accurately predict inelastic behavior of any field-constructed, building-type structural system of any complexity is dubious at best. All the analyses discussed so far such as 1st-order, 2nd-order, etc. are in essence design tools; predicting real-world behavior of complex structural systems is another matter. Once the amount of strain and structural movement needed to reach yielding has occurred all sorts of factors can have very substantial influence on the progression of the analyses.

For example, consider the materially nonlinear analysis of an irregular moment connection of one steel I-beam framing into the side of another, where the supported beam-end is bent beyond a plastic state. Modeling the I-beams using 2D-area elements (plate elements) “normally” would yield one set of results where some buckling/crippling or similar occurs and then progresses. Replicating that model but modifying the I-beam cross sections to be out of square, sloped flanges, etc. but within fabrication tolerances and the results will be very different. Adding in estimated residual stresses in the steel from the

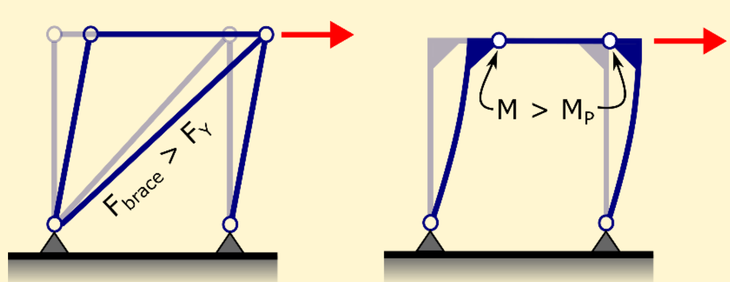


fabrication process will again change the results, probably drastically. Inclusions or mechanical damage to a member, materials with higher properties than required minima, previously endured stresses and/or reversals, specific bolt pretension values, a void or embedment in a cast-concrete member, etc. can all have substantially influential effects on results of MN analyses. There comes a point where the user can somewhat steer the results of a MN analysis using these (usually) undefinable variables, and any analysis where the user can arbitrarily manipulate the output trifles with the bounds of the scientific process.

There is a middle-ground for using 1D-line elements to anticipate some post-yielding behavior. Some software packages can accommodate force-displacement (and moment-rotation) relationships and use those instead of calculating member stresses and using a stress-strain relationship. This level of user input can allow for some local effects to be included in the derivation of the force-displacement relationship. Another approach is to use detailing, published guidance, and rigorous background analyses to predetermine the (likeliest) locations of expected plasticity, and provide nonlinear springs for post-yielding response at those locations.

Note: Using 2D-area or 3D-volume elements rather than 1D-line elements without also adopting a detailed stress-strain relationship or detailed force-displacement relationships does not allow for analyzing post-yielding material nonlinearity. Considering local effects and explicit post-yielding behavior are mutually inclusive.

A *pushover analysis* is where a structural system is (typically laterally) loaded with an increasing magnitude of load until pushover/collapse occurs. Pushover is defined as the point when the structure is rendered internally unstable upon the yielding or failure of the last moment connection, shear wall, brace, etc. able to resist loads in that direction.



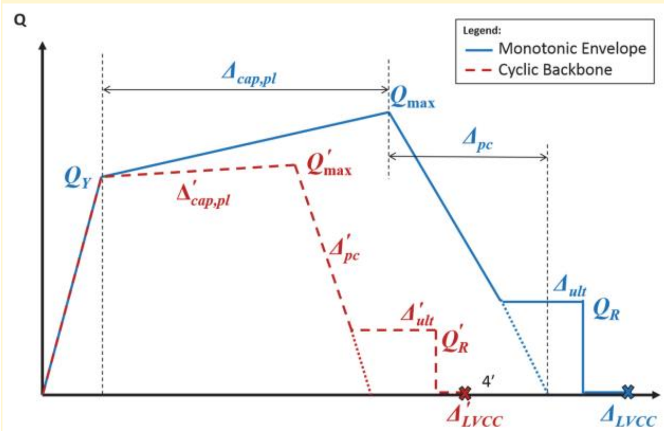
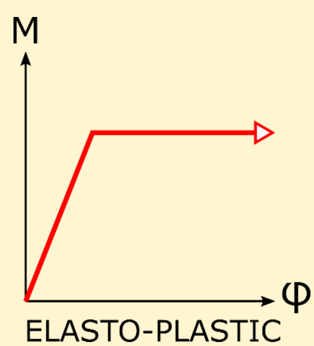


Image C: Cyclic and monotonic backbones

Pushover analyses can include material nonlinearity using 1D-line elements, but only to the extent of the response curves that can be accommodated for each member. Force-displacement (F-D) or moment-rotation relationships, or *backbone curves,* are multi-linear or numerically defined curves that can be used to explicitly include materially nonlinearity in a pushover analysis. Backbone curves are similar to stress-strain curves but can include member failure modes (cracking, crippling, etc.) in addition to material stress.



Otherwise, an elasto-plastic moment-rotation relationship would result in a pushover analysis that increases force until plastic hinging (or brace axial yielding) occurs, wherein an internal member release is added at that location and force increases continue until another hinge/brace-yielding and member release occur, and so on, until rigid-body motion finally occurs upon the collapse, or pushover. This type of pushover analysis using an elasto-plastic relationship is not truly including material nonlinearity, but is not without its merits, though, as determining the locations, quantity (redundancy), and sequence of the yield points can be beneficial.

It is no mistake that 2nd-order analyses were presented first followed by materially nonlinear 2nd-order analyses. If any part of a structure materially deforms in an inelastic manner then there will be significant enough relative structural movement from attached and nearby structure that geometric nonlinearity will be non-negligible. Simply stated, if there is MN then there will almost surely be significant geometric nonlinearity as well. Even cases like a single cantilever member concentrically loaded in axial tension would exhibit greater *true stress* than *nominal stress*, where the true stress is taken over a reduced cross-sectional area resulting from necking (Poisson effect).

Necking

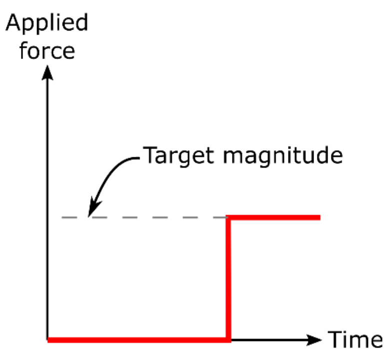
Software that can solve MN but not analyze large deflection geometric nonlinearity is often called:

MNO = Material-Nonlinearity Only

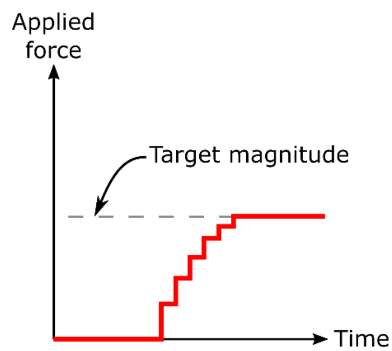
Simulation

We structural engineers often overlook how abruptly structures are loaded during a linear structural analysis. Initially, the structure is unloaded by gravity and any other externally applied forces/displacements. The next step of the analysis has the full magnitude of all the loads in place and acting, and any resulting internal forces and deformations occur instantaneously without transition. 2nd-order analyses initially perform a linear analysis using the full magnitude of loads and then slowly changes input values over subsequent iterations to find the nonlinear results that satisfy equilibrium, but load application and distortions are still thought to occur instantaneously.

If time is acknowledged and instantaneous structural response is not assumed then a step function load application like that used in linear analysis is interpreted as a shock loading, which would cause motion and dynamic response. Simulation software can apply a shock loading but it can also ramp up loads slowly over many incremental steps. The ability to slowly increase loads allows nonlinearity to be considered in an active, stepwise, converging approach rather than trying to solve for a final state all at once.



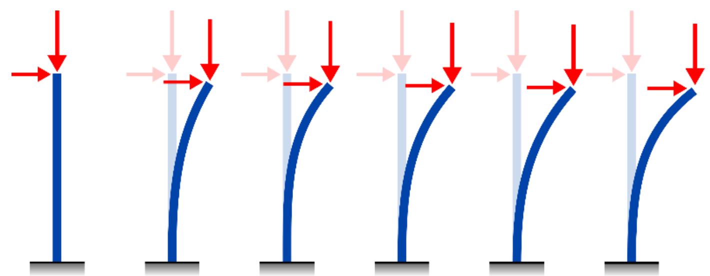
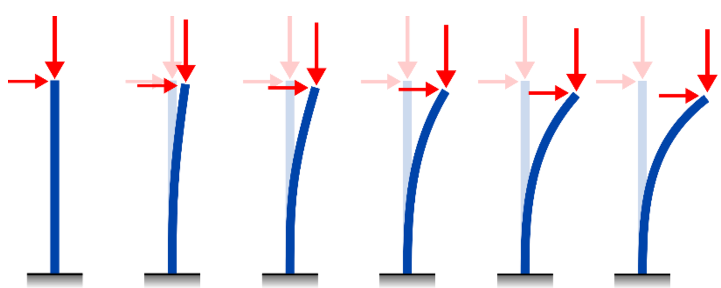
Shock loading



Incremental loading

Simulation analyses use time as a pseudo-variable used to break up loads into steps, and most simulation solvers use the principle of virtual work. Simulation uses a type of numerical analysis but it takes a different approach than the numerical analysis of 2nd-order analyses. 2nd-order analyses search for convergence from a datum, where the datum is the linear analysis results for full load magnitude, while simulation incrementally loads a structure. Geometric nonlinearity is inherently included in simulation analyses since the distorted shape is reassessed at each load increment.

Below is a pictorial representation of the difference between the iterative kinematic calculations of a 2nd-order analysis and a simulation analysis, showcasing P-Δ effects on a cantilever. Note that the cantilever deflection is exaggerated and shown scaled-up to help illustrate the concept; the deflection as shown would qualify as "large deflection" and would require a more sophisticated analysis than simulation.

2nd-order analysis		NOTE 1: this shows the result of the first iteration, which is the linear analysis result. Subsequent iterations add in geometric nonlinearity.
Simulation (2nd-order + load ramping)		NOTE 2: in this iteration step the lateral tip deflection roughly equals the result from a linear analysis, but the applied load magnitude is less than the final, target magnitude since geometric nonlinearity is included in all iterations

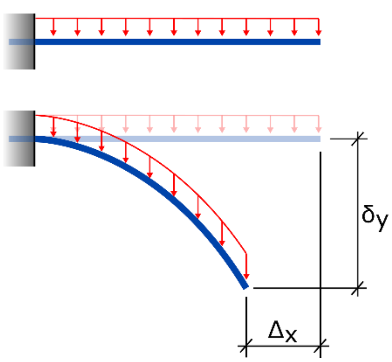
Simulation combines load ramping with the iterative kinematics of 2nd-order analyses. This layering of numerical analyses can be shown to be quite powerful in evaluating nonlinearity. If the response is so nonlinear that the software can not follow it the analysis may succeed by merely using a finer, smaller load stepping increment.

There is no official term for software that simulates structural behavior. Names like engineering simulation, mechanical simulation, mechanical event simulation, computer simulation, load ramping, and structural simulation are all common, so look for a description of software function and features for more information.

Most simulation packages and some advanced SAS can regard time more explicitly than just a pseudo-variable and can also perform dynamic analyses.

3rd-order / Large Deflection analysis

A 3rd-order analysis is another form of numerical analysis like simulation in that it often uses the principle of virtual work, but with one important difference. Simulation and 2nd-order analyses are numerical analyses iterating linear elastic or linear inelastic analyses. A 3rd-order analysis is a numerical analysis that includes nonlinear beam curvature and axial strain at each iteration. These software packages have flags or checkboxes to allow solving structural systems with *large deflections* (also commonly called *large deformations*) to a precise convergence, and typically have a library of element types with many custom suited to certain applications.



Software packages that will include full nonlinear beam mechanics and use appropriate stress and strain tensors to accommodate the large displacements and rotations typically use 2D-area and 3D-volume finite elements to do so. Software that will include the high geometric nonlinearity associated with flexible members in 1-D truss or beam elements is available but is not as common.

Another homonym?

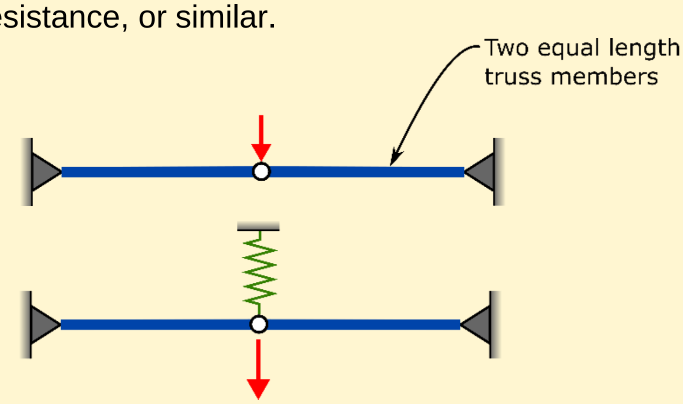
The term *3rd-order analysis* has gained popularity in recent years but it is colloquial and should not be confused with 3rd-order beam theory or with 3rd-order expressions.

Euler-Bernoulli beam theory (linear) does not recognize shear strain and shear deformations. Timoshenko presented a first order shear deformation theory for beams, which was followed by second and third order theories by Timoshenko, Rayleigh, and many others. These beam theories help avoid the inherent error in using beam theories that ignore shear strain and deformation when analyzing relatively thick or deep beams and are not directly related to 3rd-order analyses.

Describing a mathematical expression as 3rd-order indicates that the/all the primary variables are at the most raised to the 3rd power or of the 3rd derivative. This condition does not necessarily apply to 3rd-order analyses, which can encounter higher-order mathematics.

Stabilization / Rigid Motion Flag

Sophisticated analysis software will often have a feature or flag that will allow a structure or parts of a structure to move rigidly. The software will replace zeros that occur on the diagonal of the stiffness matrix with very low values, temporarily add springs (shown bottom), skip ahead to a further distorted position in search of structural resistance, or similar.



This feature allows more nonlinear situations to be analyzed, such as some snap-through type behavior and some geometrically nonlinear structural systems that are *instantaneously changeable*[5] (shown top). Instantaneously changeable structures develop structural resistance and internal forces after infinitesimal movement (e.g., vertical displacement of the middle joint).

Multi-Physics

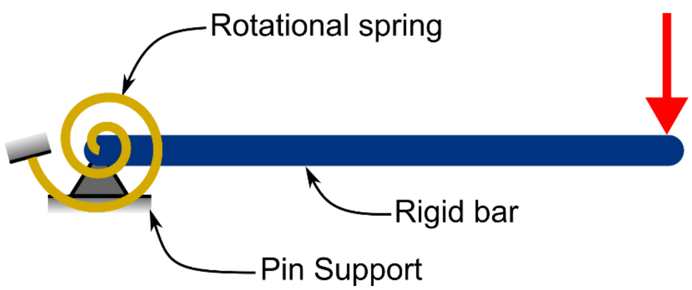
Structural simulation is just one of the many types of analyses that the most sophisticated multi-physics software can perform. High-end multi-physics simulation software can field situations involving structural behavior, dynamics, heat transfer, computational fluid dynamics, and electromagnetics, and many can include high geometric nonlinearity (3rd-order, large deflections).

How can many linear analyses solve nonlinearity?

2nd-order analyses iterate linear analyses to converge on a solution. Each analysis iteration alone cannot provide nonlinear results yet repeating those linear analyses can somehow provide nonlinear results.

The key is recursion. The result of each analysis step is in terms of itself, such as solving for a support rotation that is a function of support rotation, constituting recursion. That recursion is leveraged to compare the incoming parameter to the outgoing parameter at each analysis step to monitor convergence. The way convergence is developed can be attempted to be explained further but an example is a better vehicle for showing how iterative nonlinear analyses work.

Analysis Example – Cantilevered rigid bar with rotational spring
Included Nonlinearity: Geometric nonlinearity
Brief: A weightless rigid bar is loaded by a transverse concentrated force at its tip and elastically supported with a combined frictionless pin and rotational spring at the other end. The force is applied with respect to the global reference frame and will retain its downward vector regardless of bar rotation. The notion of needing to limit analysis to small-deflections is not applicable since the bar is rigid and there can be no beam curvature, and because the rotated position of the bar will significantly affect the location of the applied force and hence the moment at the spring, presenting geometric nonlinearity. Kinematics and a 2nd-order analysis will be used to find the bar rotation.



L = span =	200 in	5.08 m
P = applied force =	100 lbf	445 N
k_R = rotational spring stiffness =	20,000 in-lb/rad	2,260 N-m/rad

Assume that the force is applied gradually

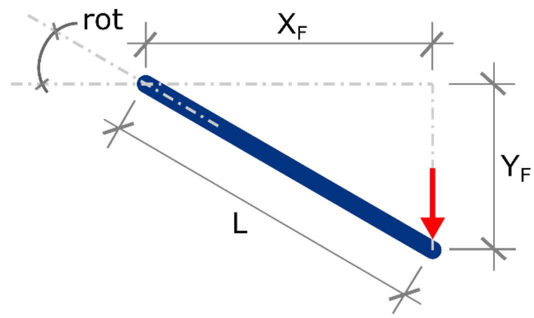
Methodology

1. Derive the kinematics of a rotated bar
2. Discuss the practical limits of this analysis and discuss the way the analysis searches for a solution while performing kinematic iterations
3. Review and discuss solutions for a few different rotational spring stiffnesses

The bar is rigid and the pin provides absolute and equal fixity in all four translation directions (upwards, downwards, leftwards, rightwards) effectively condensing this structural system to a single degree-of-freedom. The force that varies with the geometry is the bending moment since the moment arm varies with rotation of the bar.

Analysis Example – Cantilevered rigid bar with rotational spring

Derive the kinematics of a rotated bar:



$M = P \cdot X_F$

$rot = M / k_R$

$Y_F = L \cdot \sin(rot)$

$X_F = (L^2 - Y_F^2)^{1/2}$

This validity of this analysis is limited. As the initial rotation exceeds 90° the resulting position of the bar will approach straight downwards, inline with the applied force, with rotation = 90°.

$k_{R,min} = 2M_1 / \pi = 40{,}000 / \pi =$ 12,732 in-lb/rad = 1,439 N-m/rad

$k_R > k_{R,min}$ → **proceed with analysis**

Perform the 1st step of the 2nd-Order analysis:

1st:	To begin with the bar is horizontal;	
	$X_F = L$	
	$M_1 = 100 \cdot 200 = 20{,}000$ in-lb	= 2,260 N-m
	$rot_1 = 20{,}000 / 20{,}000 = 1.00$ rad	= 57.3°

This is where a linear analysis would end but iterations are obviously needed since at such a large rotation angle the moment arm of the applied force has certainly have decreased ($X_F < L$). This tendency of a decreasing moment arm with rotation is both a *stiffening mechanism*, and a simplification of a geometrically nonlinear *large deflections* effect called *apparent shortening*.

The 2nd step proceeds using the same kinematics but now the X_F value will decrease because the rotation is finite and not near-zero.

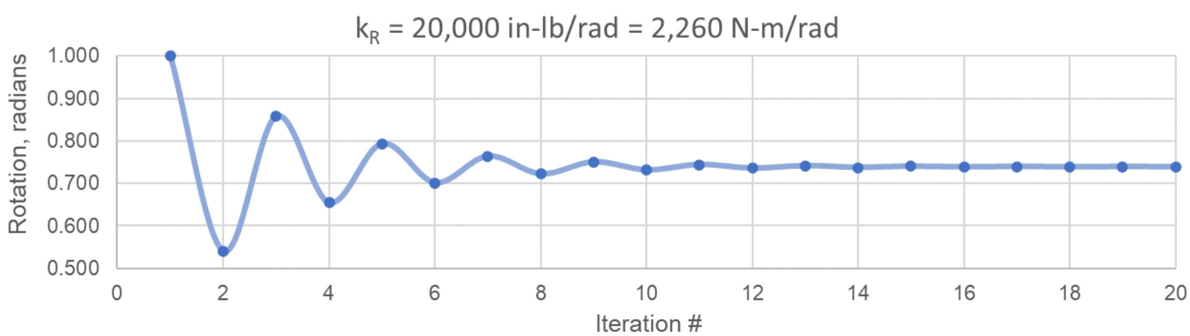
2nd:	$Y_{F1} = 200 \sin(1.00) = 168.3$ in	= 4.285 m
	$X_{F1} = (200^2 - 168.3^2)^{1/2} = 108.1$ in	= 2.745 m
	$M_2 = 100 \cdot 108.1 = 10{,}810$ in-lb	= 1,221 N-m
	$rot_2 = 10{,}810 / 20{,}000 = 0.540$ rad	= 31.0°

The change in rotation from the 1st to 2nd step is large; almost a 50% change. The 3rd step will use the rotation resulting from the 2nd step to proceed:

Analysis Example – Cantilevered rigid bar with rotational spring

3rd: Y_{F2} = 200 sin(0.54) = 102.9 in	= 2.613 m
$X_{F2} = (200^2 - 102.9^2)^{1/2}$ = 171.5 in	= 4.356 m
M_3 = 100 · 171.5 = 17,150 in-lb	= 1,938 N-m
rot_3 = 17,150 / 20,000 = 0.858 rad	= 49.1°

First 57.3° rotation, then 31.0°, and now 49.1°. It seems that the rotation values are getting swung back and forth, overcorrecting each time and making for a moving target. To avoid tediously repeating similar iterations the following graph shows the results of 20 iterations for the rotation angle in radians.

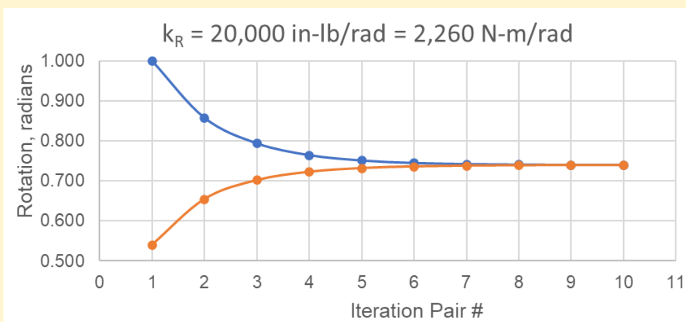


After 20 iterations the change in rotation angle is 0.05%, converging on a solution of:

0.739 rad = 42.3°

This is a fanciful problem to show how recursive analyses narrow-in on a solution and show the elusive nature of nonlinear results. The shape of the graph above illustrates how iterative numerical analyses (2nd-order, simulation, 3rd-order) converge. The analyses essentially self-determine the error and cyclically change inputs to decrease that error until the change becomes negligible, similar to how an artist using the Droste effect replicates the image smaller and smaller until it becomes insignificant.

The responses shown above and below seem harmonic but are merely tracing the analysis as it chases the moving target of a solution and are not related to time in any way nor indicative of actual harmonic response that could occur. The analyses convergence could also be shown as two converging curves.



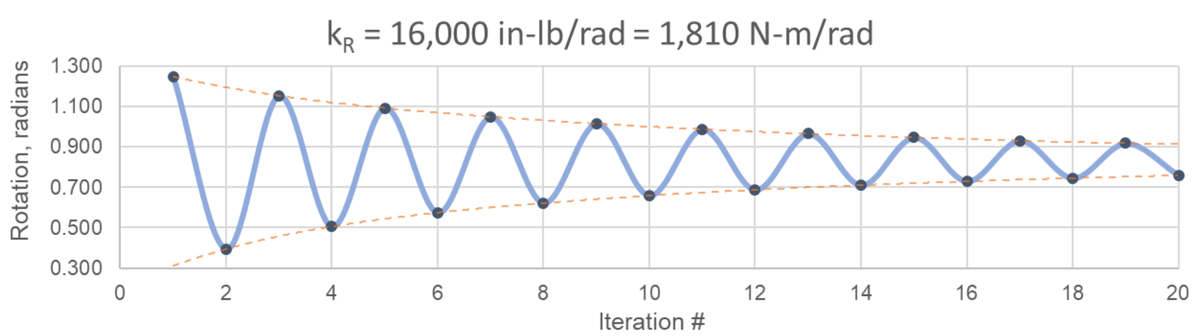
Analysis Example – Cantilevered rigid bar with rotational spring

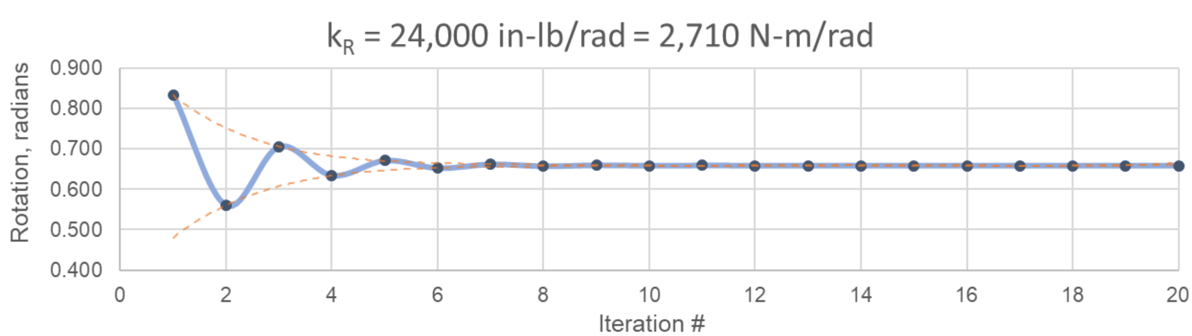
If the force were to be quickly applied and released at full magnitude then harmonic response would certainly occur as the spring swung the bar up and down, searching for a final resting position. Determining harmonic response to a sudden load application would require structural dynamics.

For reference, see the two graphs below for the same structural system with a lower and a higher rotational spring stiffness. Note how the convergence of the lower spring stiffness (more flexible) is slower, while the high spring rate (less flexible) converges quickly.

The originally analyzed spring stiffness was:

k_R = 20,000 in-lb/rad = 2,260 N-m/rad





<u>Quickly converging with a stiff structural system and needing to search at length for a result with a flexible system is a hallmark of geometrically nonlinear analyses</u>.

Toolbox

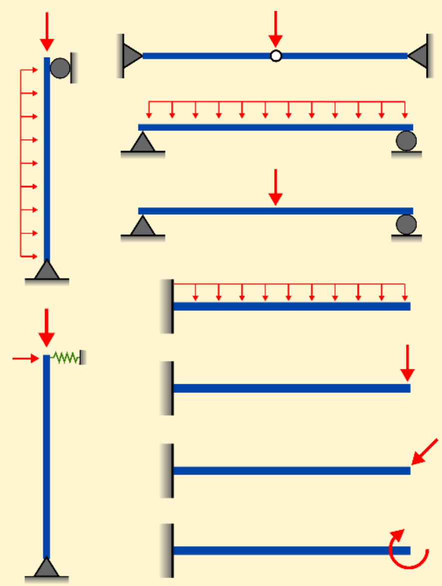
Rudimentary

Rudimentary analyses are usually limited to approximate methods or purpose-specific calculation tools with very limited scope. The nature of analyzing nonlinearities involves iterating sets of calculations and adjusting values in between sets. This limits rudimentary approaches to very simple structures such as a singular beam, simple plane frames, or structures that can be safely idealized as having only one or two degrees-of-freedom.

A 2nd-order analysis is possible to run manually and can be quite effective if not tedious and time-consuming.

Benchmark Problems

Some classic problems from history are revisited when new analysis methods or software solvers are developed. Specific structural systems or classes of structural system are carried forward to be used as a test of new methods. This legacy of revisiting certain nonlinear structure systems again and again provides *benchmarks*, showing the evolution of accuracy and efficiency in solution methods.



Approximate

There are approximate methods available for analyzing nonlinearities, with most methods pertaining to geometric nonlinearity. Some of those approximate methods have been tacitly accepted as exact methods by the engineering community due to prevalence and reasonable accuracy, like the geometric stiffness matrix. Other methods can be engineered or adjusted to refine the accuracy like discretization of members that are nonlinear by shape. Finally, there are crude methods where a linear idealization is made. These methods must be combined with engineering judgement and a firm understanding of structural behavior and should be limited to providing a rough check of proper nonlinear analyses.

One of the most prevalent approximate methods is the use of a magnification factor for moments and movement due to P-δ and P-Δ effects. This method was first published[7] by Ayrton and Perry in 1887, then popularized by Timoshenko in several of his many seminal works[8,9], and is still the basis for the alternate methods that the modern codes and standards allow use of for certain structural systems in lieu of 2nd-order analyses. This approximation is often referred to as *moment magnification*. The magnification factor is computed and then applied to 1st-order (linear) analysis results as a shortcut to obtain nonlinear results without a 2nd-order analysis.

Stephen Timoshenko "Father of Engineering Mechanics"

$$Magnification\ factor = \frac{1}{(1-\alpha)}$$

$$\alpha = \frac{P(kL)^2}{EI\pi^2}$$

The formula for alpha should look familiar. It is the reciprocal of Euler's buckling but with the actual axial compressive load in the numerator, making alpha the ratio of actual compression to the theoretical, critical buckling load. This is a simple method and can provide a viable avenue for estimating or checking P-δ and/or P-Δ response. Note that the accuracy diminishes as the ratio, alpha, approaches unity (buckling), and the guidance provided in standards will limit that ratio accordingly.

<u>Aided</u>

Most conventional SAS will include analysis capabilities for at most one or two of all the types of nonlinearity and even then, some use approximate methods instead of rigorous solvers.

> *Full nonlinearity* is sometimes used to refer to software or an analysis that includes material nonlinearity and can also analyze all geometric nonlinearity including large deflections.

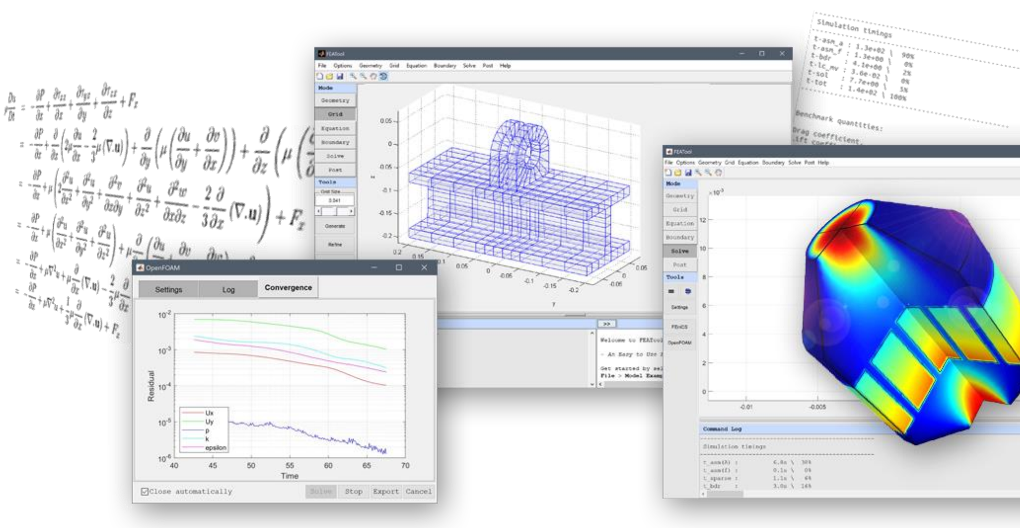


Image D: "FEATool Multiphysics FEA Simulation Toolbox Banner"

The ability of software to consider and analyze nonlinearities are often called features. The following table can be used as guide for determining the different types of nonlinearities that software can typically analyze. Note that table below is included here for informational purposes, <u>is not covered in the quiz</u>, and is subject to change over time as software innovations and complexity progresses. In this context FEA (finite element analysis) is software that only uses 2D area or 3D volume elements for static analysis and does not include kinematics features.

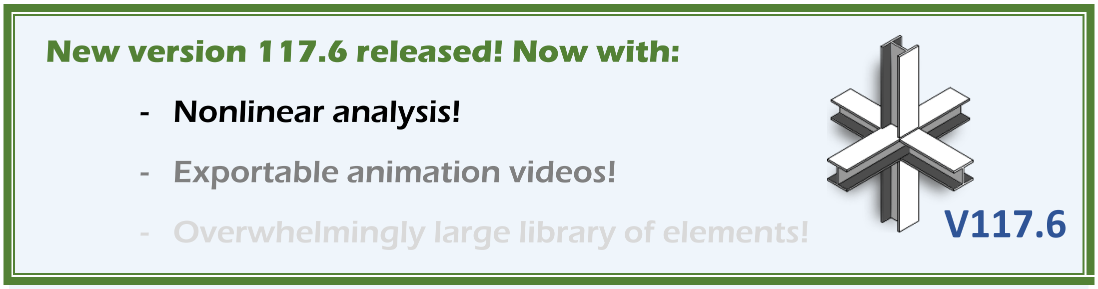
		Basic SAS	**Basic FEA**	**Advanced SAS**	**Simulation & Multi-physics**
Material	Pushover	FV	-	Y	Y
	Inelastic behavior	-	FV	FV	Y
Boundary Conditions	Gap	FV	FV	2ND	Y
	Contact	-	Y	FV	Y
	Nonuniform	-	FV	FV	FV
Directional Members		-	FV	2ND	Y
Shape		LA	Inherent for FEA	LA, FV	Inherent for FEA
Friction		-	-	-	Y
Geometric	P-δ and P-Δ	LA	-	2ND	Y
	Axial deformations	-	-	FV	Y
	Nonlinear Cables	-	-	FV	FV
	Large Deflection	-	-	-	FV (3RD)
Time	Phased/sequential construction	-	-	FV	Y
	Forces that change	-	-	-	Y

SAS	**S**tructural **A**nalysis **S**oftware
-	not included, it is not a likely feature or it would be using software beyond the practical intended use
LA	**L**inear **A**pproximation
FV	**F**eatures **V**ary, check individual software packages
Y	**Y**es, feature is typically included
2ND	yes, via **2nd**-order analysis
3RD	**3rd**-order analysis

Not shown above is a column for 3rd-order analysis software. 3rd-order analysis software will often have mostly the same features as simulation and multi-physics software, though 3rd-order analysis software may feature 1D-line elements and other features that are more in line with SAS than simulation and multi-physics.

Nonlinear Ambiguity

The lack of clarity in what nonlinear abilities SAS has is often embodied by an ambiguity issue that was mentioned at the start of the first, introductory course in this series.



Software advertisements and feature lists will often vaguely state "nonlinear analysis". What does that mean?

We do not know. We cannot know without further information.

Software providers will often use the term *inelastic* when referring to material nonlinearity, so the generic, lone term *nonlinearity* often refers to geometric nonlinearity, but what geometrically nonlinear effects are included, specifically? Ask specific questions. Ask about each type of nonlinearity and ask questions about solution methods, preset and preprogrammed functions, range of validity, etc. Focus on features that cover work that you and your company typically do, or likely could.

Conclusion

Analysis methods capable of solving structural nonlinearity can be elegant in their ability to leverage recursion and narrow-in on elusive results. However, every method and software package has limitations. Be purposeful in determining what nonlinearities are at play, which analysis methods apply, and what types of nonlinearity the software can solve.

The first steps to accommodating structural nonlinearity are defining nonlinearity, then identifying sources and instances of nonlinearity in a structural system. Those first steps were included in the first (previous) course in this series. This course presented the analysis methods that are currently available for solving structural nonlinearity and listed the types of nonlinearity that each analysis method can solve. The next course will focus on how to analyze structural nonlinearity.

References

1. McGuire W, Gallagher RH, Ziemian RD. Matrix structural analysis. 2nd ed. John Wiley & Sons; 1999.
2. Gavin H. CEE 421L Matrix Structural Analysis, lecture notes. Fall 2012. Duke University.
3. Trefethen LN, Bau D III. Numerical Linear Algebra. New York, NY: Society for Industrial & Applied Mathematics; 2000.
4. Williams A. Structural analysis: in theory and practice. Burlington: Butterworth-Heinemann; 2009.
5. Applied Technology Council. Guidelines for Nonlinear Structural Analysis and Design of Building: Part I – General. Gaithersburg: U.S. Department of Commerce; April 2017. 137 p. NIST GCR 17-917-46v1.
6. Karnovsky IA, Lebed O. Advanced methods of structural analysis. 2010th ed. New York, NY: Springer; 2009.
7. Ayrton, W. and Perry, J., 1887. On Struts. The Engineer, (62), pp.464-465, 513-515.
8. Timoshenko S. Strength of materials. - 1: Elementary theory and problems. New York, NY: Van Nostrand; 1930.
9. Timoshenko SP, Gere JM. Theory of elastic stability. 2nd ed. McGraw-Hill Companies; 1961.
10. Fertis DG. Nonlinear structural engineering: With unique theories and methods to solve effectively complex nonlinear problems. Berlin, Germany: Springer; 2010.
11. Fertis DG. Nonlinear mechanics, 2nd ed. Boca Raton, FL: CRC Press; 1998.
12. Nayfeh AH, Pai PF. Linear and nonlinear structural mechanics. Weinheim, Germany: Wiley-VCH Verlag; 2008
13. AASHTO LRFD bridge design specifications, customary U.S. units. Washington, DC: American Association of State Highway and Transportation Officials, 2012. Print.
14. Bathe, K. J. Finite Element Procedures. 2nd ed. Klaus-Jürgen Bathe, 2014. ISBN: 9780979004957

15. Klaus-Jürgen Bathe. RES.2-002 Finite Element Procedures for Solids and Structures. Spring 2010. Massachusetts Institute of Technology: MIT OpenCourseWare, https://ocw.mit.edu. License: Creative Commons BY-NC-SA.

Images

All unlisted images were created by the author or are in the public domain.

A. “Horse gallop.gif” by Alice Gee, License: CC BY 3.0
   a. https://upload.wikimedia.org/wikipedia/commons/c/cb/Horse_gallop.gif
   b. https://en.wikipedia.org/wiki/User:GeeAlice

B. “Figure 2-7: Typical local buckling of beam flanges and web in zone of plastic hinging at high levels of inelastic rotation”, NIST GCR 09-917-3, *Seismic Design of Steel Special Moment Frames*, 2009.

C. “Figure 2-5: Standard cyclic and monotonic backbones with control points”, from NIST GCR 17-917-46v1[5], pg 2-13.

D. “Banner-featool-multiphysics-matlab-fem-toolbox.png” by Precise Simulation Limited, License: CC BY 4.0
   a. https://commons.wikimedia.org/wiki/File:Banner-featool-multiphysics-matlab-fem-toolbox.png
   b. https://www.precisesimulation.com/