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WHAT EVERY ENGINEER SHOULD KNOW ABOUT ENGINEERING ECNOMIC ANALYSIS-I

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Introduction

Engineering is focused on the invention, the design, and the construction of systems and structures that improve the human condition. It does this by the application of the principles of the physical, the natural, the biological, and the social sciences in varying degrees. Part of improving the human condition is the noble notion that systems and products that result from engineering invention and discoveries should be affordable. This ultimately means that engineers look for the most economical and cost-effective ways to bring a project and idea to fruition to benefit mankind.

Managing the engineering design process requires an understanding of the economic as well as the technical ramifications of a design decision. Such decisions are made in the context of the financial, corporate, and engineering implications that attend a specific situation. However, even within this framework, different designers arrive at different decisions based on their personal preferences and their understanding of corporation's needs and/or preference space. This underscores the need to develop a unified theory of design that does not depend on individual preferences but that is built on formal design theoretic constructs. Typically, design alternatives are ranked-ordered based on their perceived measure of value or utility and the feasible choices among those alternative designs must be rational and consistent. Undoubtedly, all engineering design is performed in pursuit of benefits, frequently expressed in the form of profit to the company or in the form of benefits to the larger society. Design can be adjudged as good or bad against the benefits they offer. Under the decision-making view of engineering design, one possible objective of design decision is to maximize the benefit provided by the product. Failure to design for maximum benefits leads to products that leave room for improvement [Haz' 96].

Economic analysis essentially entails the evaluation of the costs and benefits of a binary or multiple choice or decision among alternatives. It is important to note that every decision has at least an alternative solution. Thus, the decision to do nothing when faced with a choice is in fact a decision since the decision maker made the choice to "do nothing" rather than act. The Do Nothing (DN) alternative is by default the status quo. Economic Analysis starts by ranking projects (or the possible alternatives outcomes of a project) based on economic viability to aid better allocation of resources for maximum economic benefit.

Engineering economic analysis is a combination of quantitative and qualitative techniques to analyze economic differences among engineering alternatives in selecting the preferred design. The cash flow approach is one of the major approaches in the engineering economic analysis

Engineering Economy or Engineering Economic Analysis is a specialized discipline of engineering that is focused on the economic viability and efficacy of engineering projects. Because in general, materials and resources are limited, it provides tools for systematic evaluation of the economic benefits of the alternative to the proposed solutions to engineering problems. It quantifies the costs and benefits of a project and thus gives an objective estimate of various alternate future investments in the presence of limited resources

Engineering economy falls under the larger umbrella of Decision Making which involves choices among alternatives. It involves several elements such as: problem identification, definition of the objective, cash flow estimation, financial analysis, and evaluation of alternative solutions. A structured procedure is eventually used to select the best solution to the problem taking into consideration the time value of money.

Engineering economics requires the application of engineering design and analysis principles to provide goods and services that satisfy the consumer at an affordable cost. Engineering economics is also relevant to the design engineer who must decide among several alternative materials based on technology and economic considerations. One of the overarching constructs of engineering economic analysis and indeed any economic consideration is the idea of the time value of money (TVM). TVM is the change in the amount of money over a given time period is perhaps one of the most important constructs in engineering economic analysis.

The concept of TVM is based on the idea that people would rather have money now, today, than sometime in the future. The concept of TVM implies that money is more valuable in the present rather than the future because it can earn interest which is compounded with time going forward. More specifically, TVM is an important concept in devising an investment strategy because money on hand today is worth more than the same amount promised in the future due to compounding interest and because of inflation. Again, so long as money can earn interest, TVM provides the assurance that any amount of money is worth more the sooner it is received. At the most basic level, the TVM demonstrates that, all things being equal, it is better to have money now rather than later. Thus, money on hand today can be invested and can earn interest resulting in Capital Gain (CG).

Literally speaking, time is money. The further away the time is, the less the amount of money. Hence the value of money today is not the same sometime in the future. By employing the concept of the Time Value of Money (TVM) to determine its present and future values, it is possible to distinguish between the worth of investments and the returns they produce at different times.

1.1 Difference Between Engineering Economics and Economics

Economics is the social science that describes the factors that determine the production, distribution and consumption of goods and services. Engineering economy is an engineering focus area that uses economic concepts and apply those to engineering projects.

1.2 The Major Defining Aspects of Engineering Economic Decisions

The time factor and the uncertainty fact factor are perhaps two of the most important building blocks of any project that has economic consequences or benefit. There are several types and purposes of economic decisions including (i)Cost Minimization, (ii)New Product Implementation, (iii) Process and Equipment Selection, (iv) Product and Service Improvement, (v) Product and Service Expansion, and (iv) Equipment replacement because of technology and age. Engineering design is not part of economic decision.

1.3 The Role and Function of an Engineering Economist

The engineering economist is responsible for assessing the economic viability of the organization's operations and processes and by so doing ensure its profitability and productivity from an engineering point of view because of the design background. This way the organization is positioned to compete with other organizations in the same industry and beyond.

1.4 Cash Flow

Cash flow represent the net amount of cash and cash equivalents that flows in and out of an organization. Cash received represents inflows, while cash spent represents outflows. The cash flow statement is a financial statement that shows a company's sources and usage of cash over certain periods. A company's cash flow is typically segmented into flows from operations, investments, and borrowed funds. An organization's ability to creative value and economic sustainability is contingent on its ability to generate positive cash flows and more specifically, the ability to optimize long-term free cash flow (FCF). FCF is the cash generated by a company from its normal business operations less cash expended on capital expenditures (CapEx).

In general, CapEx are funds used by an organization to acquire, upgrade, and maintain physical assets such as infrastructure, plants, buildings, technology, software, and equipment. It is used to execute new projects or investments. Capital expenditures can also be made to improve fixed assets such as roof repair, replacing an old equipment or purchasing a new one, and/or the construction of new facilities. For the most part this type of investment is made to increase the scope of operations or to add economic value to a company's operation

1.5 Rate of Return (RoR)

The rate of return (RoR) is defined as the net gain or loss of an investment over a specified time period, expressed as a percentage of the investment's initial cost. In calculating the rate of return means determining the percentage change from the beginning of the period until the end.

1.6 Discount Rate

The term discount rate can refer to either the interest rate that the Federal Reserve charges banks for short-term loans or the rate used to discount future cash flows in discounted cash flow (DCF) analysis. In a banking context, discount lending is a key tool of monetary policy and part of the Fed's function as the lender-of-last-resort. In discounted cash flow analysis, the discount rate expresses the time value of money and can make the difference between whether an investment project is financially viable or not.

1.6.1 Weighted Average Cost of Capital (WACC)

The Weighted Average Cost of Capital (WACC) is another approach used to determine the discount rate that a company uses to determine the cost of capital or to evaluate the worth of its investment. WACC is the average after-tax cost of a company's various capital sources, including common stock, preferred stock, bonds, and any other long-term debt. It is the average rate a company expects to pay to finance its assets.

Most companies operate their business using the capital raised through various sources. This includes money raised through listing shares on the stock exchange (equity), issuing interest-paying bonds or obtaining commercial loans (debt) from financial institutions. All such capital comes at a cost, and the cost associated with each type varies for each source.

$$\text{WACC}=\left(\frac{E}{V}xR_e\right)+\left(\frac{D}{V}xR_d x(1-T_C)\right)$$

$$\text{WACC} = (\text{Weighted Value of equity capital} + \text{ Weighted value of debt capital})$$

where:

E=Market value of the firm's equity
D=Market value of the firm's debt
V=E+D= Sum of Equity plus Debt
R_e=Cost of equity
R_d=Cost of debt
Tc =Corporate tax rate
E/V = the proportion of equity-based financing
D/V = the proportion of debt-based financing.

$$\left(\frac{E}{V}xR_e\right) = \textit{weighted value of equity capital,}$$
$$\frac{D}{V}xR_d x(1-T_C) = \textit{weighted value of debt of capital}$$

For example, suppose that a company obtained \$7,000,000 in equity financing by selling common shares and \$3,000,000 in debt financing. Then E/V would equal 0.7 (\$7,000,000 ÷ \$10,000,000 of total capital) and D/V would equal 0.3 (\$3,000,000 ÷ \$10,000,000 of total capital

1.6.2 Example of WACC

As an example, consider the case of NITRON Global Services, an international engineering and energy services company. The market value and the book value of the company's debt profile 'D'= \$50,000,000, with a market capitalization of (or the market value of its equity based on its most recent public fillings) 'E'= \$117,000,000.

Also, based on its recent earnings report, NITRON's cost of equity (R_e), that is, the minimum return it proposed to pay out to the shareholders after the recent shareholders' meeting is 10% (R_e=10%). The cost of debt from the company's creditors R_d is estimated at 2.5%. The prevailing tax rate is 20% based on a recent IRS pronouncements, thus T_c =20%

Hence $\frac{E}{V} = \frac{E}{E+D} = \left[\frac{\$117{,}000{,}000}{(\$50{,}000{,}000+\$117{,}000{,}000)}\right] = 0.7$

and $\frac{D}{V} = \frac{D}{E+D} = \left[\frac{\$50{,}000{,}000}{(\$50{,}000{,}000+\$117{,}000{,}000)}\right] = 0.3$

The weighted cost of equity is: $\left(\frac{E}{V}xR_e\right) = 0.7(10\%) = 0.07$

The weighted cost of debt is: $\left(\frac{D}{V}xR_d x(1-T_C)\right) = 0.3(2.5\%)(1-20\%) = 0.3(2.5\%)(80\%) = 0.006$

$$Hence: WACC = \left(\frac{E}{V}xR_e\right) + \left(\frac{D}{V}xR_d x(1-T_C)\right) = 0.07 + 0.006 = 0.076 \text{ or } 7.6\%$$

The significance and import of the Discount Rate and that of the Weighted Average Cost of Capital (WACC) cannot be overstated either for individuals or institutions. Individuals spend countless hours shopping for the best discount rate/interest rates and corporations go to great lengths to have realistic estimates for cost of capital to ensure accurate assessments of their investments to ensure shareholder value and earnings. So, this is an important element of the economic decisions in the effort to assess and determine viable economic alternatives. Some salient point about WACC include:

- WACC is the discount rate that a company uses to estimate its net present value and it is one of the metrics used by a company to determine the cost to borrow funds.
- When analyzing the potential benefits of taking on projects or acquiring another business, WACC helps the company to focus on its debt-to-equity relationship.
- If a company is considering a merger that will generate a return higher than its cost of capital, then the merger is a good choice for the company, other factors being equal. However, if the company anticipates a return lower than what its own investors are expecting, then the company should look for a different investment option.
- The cost of capital is an important parameter in assessing a firm's potential for net profitability since a majority of businesses borrowed funds to run their business.
- Under normal circumstances, a lower WACC is an indication that the business is able to attract investors at a lower cost. By contrast, a higher WACC usually means higher risk for investors which then creates the need to compensate investors with higher returns.

1.7 Earnings

A company's earnings are its after-tax net income. This is the company's bottom line or its profits. Earnings are perhaps the single most important and most closely studied number in a company's financial statements. It shows a company's real profitability compared to the analyst estimates, its own historical performance, and the earnings of its competitors and industry peers. Earnings are the main determinant of a public company's share price because they can be used in only two ways: They can be invested in the business to increase its earnings in the future, or they can be used to reward stockholders with dividends.

Value And Utility

Certain things such as joy and pain have intrinsic values whereas things such as money possess extrinsic value. Utility, on the other hand, considers the usefulness of things in general. Utility is objective and value is subjective. Utility and not Value, is "the supreme principle of all economy activity." The economic goal of civilization is to turn the natural environment from a relationship of hostility or indifference into a relationship of utility. Certain things we have from nature are without money and without price, and the effort of the industrial world is in the direction of bringing all goods nearer to that state of nature of "without money and without price" as much as practicable. Indeed, some of the necessities of life have already been brought so nearly to that condition that states and municipalities occasionally pay the small extra price so that they can distribute the necessities to citizens almost free of charge just as nature gives us rain and sunshine without charge. Thus, the effort to improve production generally is nothing other than the effort to multiply utilities and, as a consequence, to reduce their price.

Value can be looked at as the calculation form of utility. This makes sense if we consider how impossible it is to estimate the utility of a harvest, but on the other hand how easy it is to estimate its amount and its price. In any situation where value and utility are in conflict, then utility must have the upper hand.

2.1 Key Differences between Value and Utility

Utility cannot be expressed in dollars as can market value. Utility is personal because what has utility for an individual might not have the same utility for another individual. Utility is the capacity to be useful and provide satisfaction and must be present or required for something to have value. The utility of a good or service may vary form one person to the next. A good or service does not have to have utility for everyone, only utility for some. A gulf player may assign a high level of utility to a new set of golf clubs, whereas a dancer may assign higher utility to a pair of dancing shoes.

Market value, on the other hand, is aggregated and impersonal. It is what it is and cannot change because of an individual or their preferences. Market value refers to a worth that can be expressed in dollars and cents. The unexplained paradox of value is that sometimes some necessities have little monetary value, whereas some non-necessities have a much higher value. Scarcity is required for value, but scarcity alone isn't enough to create value.

In engineering economic analysis, it is important to be able to distinguish between the value of a resource and the utility (or the economic utility) of that resource. Economic ***utility*** may be defined as the amount of satisfaction experienced upon the consumption of a product or service. Utility of resource is the perceived value of the resource. It is not the original value of the resource. In other words, it is a measure of the level of fulfillment needed to satisfy a particular need or want. Economic value is the benefit that can be achieved by the purchase of the goods or use of a particular service. The value of the resource is the measurement of the benefit provided by the resource or the benefit that is gained from the use of the resource.

Based on these definitions, it can be surmised that utility is personal and subjective. It has different values for different individuals. On the other hand, market value (typically expressed in dollars) is aggregated and impersonal. The following are the 7-step of systematic economic analysis technique (SEAT)

1. Identify the investment alternatives.
2. Define the planning horizon.
3. Specify the discount rate.
4. Estimate the cash flows.

5. Compare the alternatives.
6. Perform supplementary analyses.
7. Select the preferred investment.

Classification of Cost

Cost refers to the value sacrificed with the aim of gaining something of economic value in return. The term cost itself refers to the economic value of expenditures for raw materials, equipment, supplies, services, labor, products, research, logistics, building, labor, marketing, supply chain, communication, management, and every imaginable expense required to get the business operational. Cost is the basis of profit determination for an organization It is the figure that is displayed as an expense in the company books. Every business requires an infusion of capital or funds and thus there is a need to keep a record of the funds expended on the various operations to get the business running.

Cost classification is the logical process of categorizing the different costs involved in a business process according to their type, nature, frequency, and other features to fulfil accounting requirements and to facilitate economic analysis. The classification of cost is based on the nature of the expenditure, the functions, traceability of the cost, and cost normality (namely how usual/typical or normal is the cost).

3.1 Classification by Nature

This classification of cost is based on the nature of the expenditure and can delineated into three broad categories, namely, labor cost, material cost, and expenses.

i). **<u>Material Cost</u>**: These are the costs of materials such as spare parts, and raw material, and subassemblies that are used in the production of goods. They also include all the expenses of packaging and value addition..

ii). **<u>Labor Cost</u>**: This cost includes the salary and wages paid to temporary and permanent employees for the production of the goods.

iii). **<u>Expenses</u>**: This includes all other expenses associated with the production activities, marketing, and selling the services or goods.

3.2 Classification by Functions

This is the classification based on the functional characteristics of the costs. The classification by function is based on the nature of the managerial activities such as production, administration, marketing/selling, etc.

i). **<u>Production Costs</u>**: These costs are related to the real construction or manufacturing of the goods.

ii). **<u>Commercial Costs</u>**: This cost includes the operation of a corporation less the cost of manufacturing. It consists of the management/administration, distribution and marketing/selling, etc.

3.3 Classification by Traceability

This cost is broken into direct and indirect costs. This classification is based on the degree of traceability of the cost to the final product.

i). **Direct Costs**: These are the costs that are easily related to a specific cost unit. The most significant examples are the materials used to manufacture a product or the labor involved in the production process.

ii). **Indirect Costs**: These costs are used for many purposes, which are between many cost centers or units. So, we cannot put them to one specific cost center—for instance, the rent of the place or the manager's remuneration. We will not be able to identify how to estimate costs to a specific cost unit.

3.4 Classification by Normality

This classification is based on the costs as the normal costs and abnormal costs.

i). Normal Costs: The cost of production and are the type of costs that the organization incurs at the standard level of output under normal conditions.

ii). Abnormal Costs: These costs do not occur at a particular level of output as in cost. It can be defined as a cost incurred by management that was not planned. They cost occurs outside the formal plan or budget of the business.

3.5 Types of costs

In general, the expense for producing a final product is known as Production cost. All other expenses are referred to as commercial costs. Some also refer to these costs into direct costs and indirect costs.

A good understanding of how costs behave would help the engineer in many valuable ways, especially when it comes to choice among different alternatives. Additionally, knowledge of the different types of costs that an organization generates in the process of a production activity helps in making a deciding about the appropriate mix and volume of products that would result in the greatest profit for his company.

3.5.1 First Cost

First Cost means the "cost of goods sold and refers the price paid to the manufacturer for that product and including shipping, handling, insurance, taxes, duties, or customs charges. It is the total cost per unit as invoiced by the manufacture of a product.

3.5.2 Fixed, Variable, and Mixed Costs

Based on behavior, costs can be categorized as fixed, variable, or mixed. Fixed costs are constant regardless of activity level, variable costs change proportionately with output, and mixed costs are a combination of both. Understanding how these costs behave helps management to plan ways.

i). Variable costs are those costs that vary with the amount of activity. Examples of variable costs are direct material cost, direct wages, direct expenses, etc. For example, the amount of cement that the Block molding company-- BMOLD uses to mold blocks would vary depending on the number of blocks that the company molds in a month. Similarly, BMOLD would incur more direct labor costs as the quantity of blocks molded increases. The variable cost per block helps to determine the cost for different volumes of production in other to satisfy the high demand for cement blocks in the area.

ii). **Fixed costs** do not change with the amount of activity. An example of a fixed cost for BMOLD is the monthly rent on the space where the block molding activities take place. The rent would be the same whether he molds 20 blocks or 30,00 blocks in the facility. It is worthy of note that fixed costs are not absolutely fixed for all time. In fact, fixed costs are fixed only in relation to a particular level of production capacity

iii). **Mixed costs** contain elements of both variable and fixed costs. A good understanding of mixed cost is important in order to predict how costs will change with different levels of production. Typically, a portion of a mixed cost may be present in the absence of all activity (e.g., electricity usage when no production activity is taking place). However, the cost may also increase as activity levels increase. As the level of usage of a mixed cost item increases, the fixed component of the cost will not change, however, the variable cost component will increase. The formula for this relationship is:

$Y = F + Vx$

Y = Total cost,

F = Total fixed cost,

V = Variable cost per unit of activity,

x = Number of units of activity

3.5.3 Marginal and Incremental Costs

. **Marginal cost**s refer the change in total costs per unit change in output. In marginal cost, the focus is on the increased total cost that will arise from the production of one more unit of output. For example, the block molding company BMOLD currently molds 5,000 blocks at a cost of $5,500, so that the average cost per block $1.10. However, if the production line makes 5,001 units, the total cost is $5,501, so that the marginal cost of the one additional unit is only $1. This is to be expected because there is hardly any additional overhead cost associated with a single unit of output, thus resulting in a lower marginal cost.

Incremental cost is the extra cost that a company incurs if it manufactures an additional quantity of units. For example, consider a company that produces 100 units of its main product and decides that it can fit 10 more units in its production schedule. The additional cost it will incur for producing these 10 units is the incremental cost. Thus, incremental costs often refer to the difference between alternatives. For example, a portable Wi-Fi company is considering increasing production because of increased overseas demand.

The current cost configurations are as follows

1. Currently, 200,000 units of portable Wi-Fi has a total production cost of $4,0000,000 or $20/unit ($4m/200,000 =$20/unit)
2. The new configuration under consideration is as follows: 350,000 units of the same product with an anticipated production cost of $5,200,000 or $15/unit ($5.2m/350,000=$15/unit)

Hence the total incremental cost to produce the additional 150,000 units is $1,200,000 (or $5.2m-$4m) and so the incremental cost per unit is $8 ($1,200,000/150,000)

Incremental cost is sometimes confused with marginal cost, and while they are closely related, there is a difference between the two. While marginal cost refers to the change in total cost resulting from producing an additional unit of output, incremental cost refers to the total additional cost associated with the decision to expand output or to add a new product line. Marginal cost considers the increased total cost that would result from the production of one more unit. However, incremental cost considers only the total costs that result from the decision to produce extra units and so it has a relatively wider connotation.

Both are concerned with the change in the total cost with marginal costs focusing on the increase or decrease in costs that results from producing an additional unit of output while incremental cost is concerned with the change in the total output as a result of the change in the methods of production or addition of a product line or sales channel, or the use of improved technology.

3.5.4 Sunk Costs

Sunk costs refer to funds that have already been spent and so are irrecoverable. A company may have a number of sunk costs, such as the cost of advertising and marketing, salaries, salaries, insurance, rent, nonrefundable deposits, or repairs, and equipment (so long as each of these items cannot be recovered. For individuals this could include the school or training fees, haircuts, food, rent, transportation, etc. For these types of costs, once made can never be recovered.

Sunk costs are typically not included as part of the consideration in future decisions, as they are seen as irrelevant and so do not affect current and future budgetary outcomes. Thus, when making business decisions, only relevant costs should be considered, especially the future costs whose funds are yet to be committed or incurred. To make an informed decision, a business only considers the costs and revenues that would change as a result of the decision at hand. Because sunk costs do not change, they should not be considered. Sunk costs are excluded from future business decisions because they will remain the same regardless of the outcome of a decision.

It is important to understand the notion of sunk cost because of its psychological ramifications (also known as "Sunk Cost Fallacy") on individuals and organizations. The sunk cost fallacy is a psychological barrier that tend to tie people to unsuccessful endeavors due to the fact that they have already committed resources to it. The sunk cost fallacy leads to an improper mindset that a corporation or an individual may develop when exploring different alternatives and options. This fallacy is propagated under the notion (albeit mistaken) that committing to a position is somewhat justified because resources have already been committed to such a plan or position. This could result in improper long-term strategic planning and attendant decisions that are based on costs that are no longer relevant.

Interest Formulas and Time Value of Money

Time value of money is based on the idea that people would rather have money now, today, rather than sometime in the future. Money is more valuable in the present rather than the future because of it can earn interest which is compounded with time going forward. More specifically, Time Value of Money (TVM) is an important concept in devising an investment strategy because money on hand today is worth more than the same amount promised in the future due to compounding interest and because of inflation. Again, so long as money can earn interest, TVM provides the assurance that any amount of money is worth more the sooner it is received. At the most basic level, the time value of money demonstrates that, all things being equal, it is better to have money now rather than later. Thus, money on hand today can be invested and can earn interest resulting in capital gain (CG).

TVM has two major components, namely, i). **Present Value(P)** and ii). F**uture Value (F)**.

4.1 Interest and Interest Formulas

The word interest in the context of engineering economics means the extra amount earned by the investor along with the investment (or) the amount owed by the borrower along with the amount borrowed. It is the monetary charge for the privilege of borrowing money, typically expressed as an annual percentage rate (APR). Interest is the amount of money a lender or financial institution receives for lending out money. In the broader area of economics, interest can also refer to the amount of ownership a stockholder has in a company, usually expressed as a percentage. For our purposes, Interest is the cost of borrowing money, where the borrower pays a fee to the lender which is the cost of the loan. On the other hand, it can be viewed as the benefit of an investment in the form of extra amount earned from the investment. In general, there are two types of interests, namely, simple interest and compound interest.

Simple interest is based on the principal amount of a loan or the deposit in a savings account. Simple interest doesn't compound, which means a creditor will only pay interest on the principal amount and a borrower would never have to pay more interest on the previously accumulated interest. Simple interest is calculated only on the principal amount of a loan or deposit, so it is easier to determine than compound interest.

Compound interest (also known as compounding interest) is the interest on a loan or deposit calculated based on both the initial principal and the accumulated interest from previous periods. It is based on the principal amount and the interest that accumulates on it in every period. The idea of compound interest is believed to have originated in Europe in the 17th century. Compound interest can be thought of as "interest on interest and makes an amount grow at a faster rate than in the case of simple interest. The rate at which compound interest accrues depends on the frequency of compounding. The higher the number of compounding periods, the greater the compound interest. For example, the amount of compound interest accrued on $1000 compounded at 8% annually will be lower than that on $1000 compounded at 8% semi-annually over the same time period. Compound interest is calculated on the accumulated principal and interest and hence it is different for every span of time period as it is calculated on the amount not the principal. Define the following:

Let P = *principal*

i = *nominal annual interest rate in percentage terms*

n = *number of compounding periods*

F= *future value after interest has been earned*

The compound interest is the total amount of principal and interest in future (or future value) minus principal amount at present (or present value). Let t=1 to n

Then, $For\ t = 1, F_1 = P + Pi = P(1 + i)$

$For\ t = 2,$

$F_2 = P(1 + i) + i[P(1 + i)] = P(1 + i)(1 + i) = P(1 + i)^2$

$For\ t = 3$

$F_3 = P(1 + i)^2 + iP(1 + i)^2 = P(1 + i)^2[1 + i] = P(1 + i)^3$

$For\ t = n$

$F_n = P(1 + i)^{n-1} + iP(1 + i)^{n-1} = P(1 + i)^{n-1}[1 + i] = P(1 + i)^{(n-1)+1} = P(1 + i)^n$

Compound Interest earned, CI= $F - P \Rightarrow P(1 + i)^n - P = P[(1 + i)^n - 1]$

The **compounding Factor** = $[(1 + i)^n - 1]$

Problem 4.1: Assume there a Savings deposit of $20,000 for 4 years at an interest rate of 10% compounded annually. What is the compound interest and what is the compound interest factor.

Compound Interest CI= $P[(1 + i)^n - 1] = \$20{,}000[(1 + 0.1)^4 - 1] = \$20{,}000[1.4641 - 1]$

So, the compound interest earned in 4 years is $= \$20{,}000[0.4641] = \$9{,}282.00$

The compound factor =$[(1 + i)^n - 1] = [1.4641 - 1] = 0.4641$

Problem 4.2

A sum of $5,000 (P=$5,000) was deposited for 5 years at a rate of 15% (this means annual rate is 15%). Use a table to show the difference in computation between simple interest and compound interest on this principal. See the solution in table 1.

Simple Interest Calculation (r = 15%)	Compound Interest Calculation(r = 15%)
For 1st year: P = $5,000 Duration = 1 year, Interest = $750 ($5000*0.15)	**For 1st year:** P = $5,000 Duration = 1 year, Interest = $750 ($5000*0.15)
For 2nd year: P = $5,000 Duration = 1 year, Interest = $750	**For 2nd year:** P = $5,750 ($500+$750) Duration = 1 year, Interest = $862.50($5750*0.15)
For 3rd year: P = $5,000 Duration = 1 year, Interest = $750	**For 3rd year:** P = $6612.50($5750+$862.50) Time = 1 year, Interest = $991.875 ($6612.0*0.15)
For 4th year: P = $5,000 Duration = 1 year, Interest = $750	**For 4th year:** P = $7,604.375($6612.5+$991.85) Duration = 1 year, Interest = $1,140.656
Total Simple Interest = $3,000	Total Compound Interest = $3,745.03
Total Amount = $5,000 + $3,000 = $8,000	Total Amount = $5,000 + $3,745.03=$8,745.03
Table 1: Computation of Simple Interest vs Compound interest for the same rate principal and time	

Problem 4.3

Ronald needs some money ($10,000) to start an online business. Rather than go to the bank because of his credit score, he approaches his childhood friend Donald who agrees to lend him the $10,000 at an int i). How much is this transaction worth to Donald. In other words, how much does Donald make from this transaction after 3 years based on period-by-period computation at interest of 20% compounded semiannually for a period of 3 years

Time Period	Amount Calculation
1st half of year 1	Principal = $10,000 Interest = 10% × $10,000 = $1,000 Amount = $10,000 + $1,000 = $11,000
2nd half of year 1	Principal = $11,000 Interest = 10% × $11,000 = $1,100 Amount = $11,000 + $1,100 = $12,100
1st half of year 2	Principal = $12,100 Interest = 10% × $12,100 = $1,210 Amount = $12,100 + $1,121 = $13, 310
2nd half of year 2	Principal = $13,310 Interest = 10% × $13,310= $1,331 Amount = $13,310.0 + $1,331.0 = $14,641.00
1st half of year 3	Principal = $14,641.00 Interest = 10% × $14,641 = $1,464.1 Amount = $14,641.0 + $1,464.1 = $16,105.1
2nd half of year 3	Principal = $16,105.10 Interest = 10% × $16,105.10 = $1,610.51 Amount = $16,105.10+ $1,610.51 = $17,715.61

Table 2: Computation of Computation of Compound Interest

From the problem description: $r_a = 20\%, \Rightarrow\Rightarrow r_{sa} = \left(\frac{20}{2}\right)\% = 10\%, n = 3, m = 2, P = \$10{,}000$

Solution: i). $A = \$17{,}715.61$

ii). **Semiannual /Half Year (m=2, t=3)**

$$A = P\left(1 + \frac{r_a}{2}\right)^{2n} = \$10{,}000\left(1 + \frac{0.2}{2}\right)^{2(3)} = \$10{,}000(1 + 1.1)^6 = \$10{,}000(1.771561) = \$17{,}715.61$$

4.2 Nominal and Effective Interest Rates

The **nominal interest rate** (NIR) is the stated interest rate of a financial instrument such as a loan or investment and signifies the actual monetary price borrowers pay lenders to use their money or the money that accrues t the investor. If the nominal rate (also called the coupon rate) on a loan is 10%, then the debtor can expect to pay $100 of interest if the amount of the loan is loaned $1000.

The nominal interest rate (NIR or simply r) is the annual interest rate that does not consider the effect of compounding of the interest or the discount rate. It is also known as the simple, announced or stated interest rate. Please note that in practice, the announced or stated interest rate is the nominal annual rate. If it is not annual then a different period would have to be specified. So, a stated interest rate of 15% compounded monthly, says that the annual nominal rate is 15% but compounded monthly. Please note that if the period of interest for the stated interest is not specified, then it is assumed to be the **annual rate**

The **effective interest rate**, also known as the effective annual rate (EAR) is the interest rate which takes compounding into account during the year. The effective interest rate is the actual percent interest that a borrower pays on their loan or earns on their investment. It is the actual annual interest considering compounding

If the interest is compounded annually, the normal interest rate and the effective interest rate will be the same. In other words, if there is compounding, the effective interest rate is higher than the nominal rate. If the rate is compounded once a year, then the rates would be the same since the compounding is once. If the interest is compounded more than once a year, then the effective interest rate will be higher than the nominal rate. The effect of the more frequent compounding is that the effective interest rate per year is higher than the nominal interest rate. So, the effective interest rate for daily>monthly>quarterly>semi-annual>annual.

As an example, suppose the savings account policy of a bank says as follows: "12 % interest rate compounded monthly." In this case what is the interest i). in the first month, and ii). how much would the total amount be at the end of one year if the amount deposited is $500.

A few things to note about this simple example. The 12 % interest is for one year since there is no period qualifier. Of course, if the interest is not for one year then it would be necessary to state the period of interest.

i. Compounded monthly indicates that there are 12 interest periods per year (m=12), with an interest period being 1 month.
ii. Since we have 12 interest periods (due to 12 month in a year), then the interest rate per interest period (1 month in this case) is 1% (12÷12). For the one-year duration, we have 12 interest periods.

Problem 4.4 Data: P=principal=$500, i=1%, n =1*12=12, F=future value

$F = P(1 + i)^m$

For 1 month, m=1, $F = P(1 + i)^1 = \$500(1.01)^1 = \$505$, thus 1 month interest =$5 (505-500)

For 1 year, m=12

$F = P(1 + i)^m = \$500(1.01)^{12} = \$500(1.1268) = \$563.412$. So, the accrued interest on the $500 is $63.412 in one year.

Thus, the interest paid is $[(1.1268) - 1] = 0.1268\ or\ 12.68\%$. which is the effective interest rate.

Alternatively, the interest can be computed as follows $[\$(63.412) \div \$500] = 0.1268 = 12.68\%$

4.2.1 Computation of Effective Interest Rate and Nominal Interest Rate

Let *r = the nominal interest rate (in percentage) for the interest period (typically one year horizon)*
m = the number of interest compounding periods/ subperiods per year
i_a = the effective interest rate for the period of interest
n = number of years or number of annual periods

Let the nominal interest rate compounded quarterly $=r$, then the equivalent quarterly interest rate is $\frac{r}{m}$. The future amount at the end of the year is: $F = P\left(1+\frac{r}{m}\right)^m$. We showed that this amount is as a result of compounding which yields the effective interest rate.

Let $\mathrm{i_a}$ be the effective annual interest rate per year in where (n=1)periods,

Then: $F = P(1+i_a)^n = P(1+i_a)^1$

Since the two quantities are identical, $P(1+i_a)^1 = P\left(1+\frac{r}{m}\right)^m \Rightarrow\Rightarrow P(1+i_a) = P\left(1+\frac{r}{m}\right)^m$

Cancel P on both sides, then: $(1+i_a) = \left(1+\frac{r}{m}\right)^m \Rightarrow i_a = \left[\left(1+\frac{r}{m}\right)^m - 1\right]$ …..equation 4.1

Hence <u>(EAR), the effective interest rate</u>: $i_a = \left[\left(1+\frac{r}{m}\right)^m - 1\right]$, where *r* is the nominal rate.

But we know from equation 4.1, that: $(1+i_a) = \left(1+\frac{r}{m}\right)^m \Rightarrow\Rightarrow \left(1+\frac{r}{m}\right)^m = (1+i_a)$

$Hence$: $\left(1+\frac{r}{m}\right)^m = (1+i_a) \Rightarrow\Rightarrow \left[\frac{r}{m} = (1+i_a)^{\frac{1}{m}} - 1\right]$

Therefore, $\frac{r}{m} = \left[(\mathbf{1}+\boldsymbol{i_a})^{\frac{1}{m}} - \mathbf{1}\right]$, m=number of compounding periods

Hence the annual discount rate r, given the effective annual interest rate $\boldsymbol{i_a}$: $r = \boldsymbol{m}\left[(\mathbf{1}+\boldsymbol{i_a})^{\frac{1}{m}} - \mathbf{1}\right]$

<u>NOTE</u>: The monthly nominal or monthly discount rate can be obtained directly using the effective annual rate or the annual nominal rate as follows:

i. Monthly discount rate(based on effective annual interest rate)= $\left[(1+i_a)^{\frac{1}{12}} - 1\right]$
ii. Monthly discount rate based on annual nominal/discount rate=Divide the nominal by 12.

<u>Problem 4.5</u>

Assume the effective annual interest rate (EAR) is i_a =9%, is compounded quarterly (m=4)

i). what is the nominal interest rate, ii) what is the monthly nominal/discount rate

i). Nominal annual rate $r = m\left[(1+i_a)^{\frac{1}{m}} - 1\right] = 4\left[(1+0.09)^{\frac{1}{4}} - 1\right] = 0.0871 = 8.71\%$

Recall that the nominal interest rate **<u>is always less</u>** than the effective interest rate

ii). The monthly nominal rate based on effective rate=$\left[(1+0.09)^{\frac{1}{12}} - 1\right] = 0.0072 = 0.72\%$ **<u>OR</u>**

ii). The monthly nominal rate: Divide the nominal in i). by 12, that is $(0.087 \div 12)\% = 0.72\%$

<u>Problem 4.6</u>

Reverse the question in problem 4.5 for confirmation. If the nominal effective interest rate is 8.71% compounded quarterly (m=4) that is r =0.0871, what is the effective annual interest rate.

EAR: $\boldsymbol{i_a} = \left[\left(\mathbf{1}+\frac{\boldsymbol{r}}{\boldsymbol{m}}\right)^{\boldsymbol{m}} - \mathbf{1}\right] = \left[\left(\mathbf{1}+\frac{\mathbf{0.0871}}{\mathbf{4}}\right)^{\mathbf{4}} - \mathbf{1}\right] = \mathbf{0.0899} \approx \mathbf{9.0\%}$.

Problem 4.7

The nominal interest rate is 8% compounded semiannually, what is the effective annual interest rate i_a

r = *8%, compounded semiannually, semiannual rate* $=r/2$

i_a = *the effective interest rate per year given discount period m=2*

The effective (actual)annual interest rate earned per year is given by:

$$\text{EAR}= i_a = \left[\left(1+\frac{r}{m}\right)^m - 1\right] \Rightarrow i_a = \left[\left(1+\frac{0.08}{2}\right)^2 - 1\right] \Rightarrow [i_a = (1.0816-1)] = 8.16\%$$

Note: The effective interest rate is **always greater** than the nominal interest rate.

Problem 4.8

If the nominal interest rate (r) of 8% is compounded quarterly, i). what is the effective interest rate compounded monthly, ii). what is monthly discount/nominal rate

i) First convert quarterly nominal interest rate to annual effective interest rate and then compute the effective monthly rate by dividing the effective annual rate by (EAR) by 12

iii). Compute the monthly discount or nominal rate using the effective annual rate

$$i_a = \left[\left(1+\frac{r}{m}\right)^m - 1\right] = \left[\left(1+\frac{0.08}{4}\right)^4 - 1\right] = 0.0824 = 8.24\%,$$

i). The effective annual interest rate compounded monthly :$(i_a \div 12) = (0.0824 \div 12)\% = 0.687\%$

ii). The monthly nominal rate $=\left[(1+i_a)^{\frac{1}{12}} - 1\right] = \left[(1+0.0824)^{\frac{1}{12}} - 1\right] = 0.00662 = 0.662\%$

Notice that the monthly effective rate is greater than the monthly nominal rate (0.687%>0.662%)

Problem 4.9

The bank charges an interest of 1% every month on a small loan. What is:

i). the nominal rate: r,

ii). The effective annual interest rates on the loan, i_a

iii). Verify that the nominal interest rate from your solution in ii) is accurate.

Solution: i). Note that the 1% interest rate is not the nominal rate. However, since the monthly rate is the same every month, the annual nominal interest rate will be constant. Hence the annual nominal interest rate is: $\text{NIR} = \text{r} = 12 * 1\% = 12\% = (12 \div 100) = 0.12$

i). Hence, the nominal interest rate per year is 12%

ii). the effective annual interest rate :$i_a = \left[\left(1+\frac{r}{m}\right)^m - 1\right] = \left[\left(1+\frac{0.12}{12}\right)^{12} - 1\right] = 12.7\%$

iii). $nominal\ rate:\ r = m\left[(1+i_p)^{\frac{1}{m}} - 1\right] \Rightarrow r = 12\left[(1+0.127)^{\frac{1}{12}} - 1\right] = 0.1202 \approx 12\%$

Problem 4.10

Compute the nominal interest rate compounded monthly (or the monthly discount rate) if you are charged **8.5%** compounded quarterly.

i) Convert quarterly to effective rate

ii) Convert the effective annual rate to nominal annual rate

iii). Compute the nominal monthly or periodic rate by dividing the nominal in ii). by 12.

i). The effective annual interest rate= $i_a = \left[\left(1+\frac{r}{m}\right)^m - 1\right] = \left[\left(1+\frac{0.085}{4}\right)^4 - 1\right] = 0.0877 = 8.77\%$

ii). Nominal annual interest rate : $r = m\left[(1+i_a)^{\frac{1}{m}} - 1\right] = 12\left[(1+0.0877)^{\frac{1}{12}} - 1\right] = 0.0844$

iii). Nominal monthly interest rate: $\left[(1+0.0877)^{\frac{1}{12}} - 1 = 0.7\%\right]$ OR $[(0.0844 \div 12) = 0.7\%]$

Problem 4.11

As part of its marketing strategy, a credit card company wants to be competitive with the Millennium population, especially young graduates and charges a nominal 14% interest per year, compounded monthly. What effective annual interest rate is the company charging this target population?

$r = 14\,\%\,per\ year\ or\ \ 0.14/year,\ m = 12\ months/year$

$$i_a = \left[\left(1+\frac{r}{m}\right)^m - 1\right] = \left[\left(1+\frac{0.14}{12}\right)^{12} - 1\right] = 0.14934 = 14.93\%$$

Note: It may be desired to find the effective interest rate for a period other than annual. In this case, we must adjust the period for "r" and "m" as needed. For example, if the effective interest rate per semiannual period (every 6 months) is desired, then

r = nominal interest rate per 6 months and m = number of compounding periods per 6 months

Then the effective interest rate, i_a, per semi-annual period, is: $i_a = \left[1+\frac{r}{m}\right]^m - 1$

Problem 4.12

If the nominal interest rate for 6 months is 12%. What is the effective semiannual rate: i_a

$r =$nominal interest rate semiannually =12%, $m =$ number of compounding periods per 6 months=6

$$i_a = \left\{\left[1+\frac{r}{m}\right]^m - 1\right\} = \left[1+\frac{0.12}{6}\right]^6 - 1 = 0.126 = 12.6\%$$

Problem 4.13

A bank introduced a new product for seniors for which it charges 8% interest, compounded quarterly, what effective annual interest rate is the bank charging seniors for this product?

Let m = number of quarters/year=4, hence: $i_a = \left[1+\frac{0.08}{4}\right]^4 - 1 = 0.0824 = 8.24\%$

4.3 Interest Formulas-Compound Interest (CI)

Principal: $P =$ *The present principal sum or the amount that was initially borrowed from the bank or invested.*

Rate: $\boldsymbol{r} =$ *nominal annual interest rate, the rate of interest at which the principal amount is loaned or invested.*

Time: $n =$ *the number of annual periods for which the principal amount is loaned or invested.*

Amount: $A =$ *single payment in a series of n equal payments made at the end of annual period compounding interest*

Amount: $F =$ *the future sum after compounding interest*

Compounding periods: $m =$ *number of interest compounding periods per year related*

The compound interest is calculated, after calculating the total amount over a period of time, based on the rate of interest, and the initial principal. For an initial principal P, rate of interest(in percent) per annum of *r*, time period *n* in years, the frequency, or the number of times the interest is compounded annually is *m,* then the formula for calculating of amount is as follows

1. *Annual Compounding*: $F = P(1+r)^n$
2. *Semiannual Compounding*: $F = P\left(1+\frac{r}{2}\right)^{2n}$
3. *Monthly Compounding*: $F = P\left(1+\frac{r}{12}\right)^{12n}$

In general, if there *m* compounding periods per year then: $F = P\left(1+\frac{r}{m}\right)^m$

The Compound interest in general is : $CI = [P(1+r)^n - P] = P[(1+r)^n - 1]$

4.3.1 Specific Cases of CI

For m ≠1, time period n (n≠1), r (percent %)

$$F = P\left(1+\frac{r}{m}\right)^{mn} : OR \Rightarrow\Rightarrow \quad F = P\left(1+\frac{r/100}{m}\right)^{mn}$$

1. Annual /One Year (m=1), $F = P(1+r)^n$
2. Semiannual /Half Year (m=2), $F = P\left(1+\frac{r}{2}\right)^{2n}$, $CI = \left[P\left(1+\frac{r}{2}\right)^{2n} - P\right]$
3. Quarterly (m=4), $F = P\left(1+\frac{r}{4}\right)^{4n}$, $CI = \left[P\left(1+\frac{r}{4}\right)^{4n} - P\right]$
4. Monthly (m=12), $F = P\left(1+\frac{r}{12}\right)^{12n}$, $CI = \left[P\left(1+\frac{r}{12}\right)^{12n} - P\right]$
5. Daily (m=365), $F = P\left(1+\frac{r}{365}\right)^{365n}$, $CI = \left[P\left(1+\frac{r}{365}\right)^{365n} - P\right]$

For n=1 year(annual)

If the compounding is **semiannually** then the nominal interest rate = $^r/_2$, hence $\mathrm{F} = \mathrm{P}\left(1+\frac{\mathrm{r}}{2}\right)^2$

If the compounding is **quarterly**, then the nominal interest rate = $^r/_4$, hence :$F = P\left(1+\frac{r}{4}\right)^4$

If the compounding is **monthly**, then the nominal interest rate is = $^r/_{12}$, hence: $F = P\left(1+\frac{r}{12}\right)^{12}$

if the compounding is **daily**, then the nominal interest rate is = $^r/_{365}$, $F = P\left(1+\frac{r}{365}\right)^{365}$

4.3.2 Important Notes About Interest Rates

Note: If the interest is compounded annually, the nominal interest rate and the effective or periodic interest rate will be the same. If the interest is compounded more than once per year, say semi-annually, quarterly, or monthly, then the effective or periodic interest rate would be higher than the original nominal annual interest rate. Please note the following distinctions as they are important in understanding interest rates. Examples: "12% interest" means that the interest rate is 12% per year, compounded annually. "12% interest compounded monthly" means that the interest rate is 12% per year (not 12% per month), compounded monthly. Thus, the interest rate is 1% (12% / 12) per month.

Problem 4.14

If a lender charges 15% interest, compounded monthly, what is the effective interest rate per quarter?

Hint: The nominal interest rate per year $r = 15\%$, so interest rate per month $=\frac{15\%}{12} = 1.25\%$
m = number of compounding periods per quarter (no of months in a quarter =3)
Let the nominal interest rate per quarter $= r_Q$, then: $r_Q = 3(1.25\%) = 3.75\%\ OR$ (15%÷4)=3.75%
Thus, the effective interest rate/quarter $i_a = \left\{\left[1 + \frac{0.0375}{3}\right]^3 - 1\right\} = 0.03797 = 3.80\%$

For Problem 4.14, above please note as follows: The nominal interest per quarter =3.75%, while the effective interest rate per quarter =3.8%. This result is consistent with the idea that the effective interest rate is always greater than the nominal interest rate.

The Gradient Factor as Interest Factor

The idea of economic gradients grew out of the study of equipment replacement strategies due to deterioration and obsolescence which occurred at constant fixed rate or at linearly increasing constant rate. Thus, it is associated with the constant or linearly increasing amount accumulated deterioration and in the case of capital budgeting, it is associated with accumulated deterioration or accumulated growth.

In capital budgeting, annual or periodic payments occur in equal payment series in which case the series of uniformly increasing payments such as $400 (occur at the end of year 1 or first period), $800(occur at the end of year 2 or second period), $1200 (occur at the end of year 3 or third period),… and $400n (occur at the end of year n or nth period). Similarly, a uniformly decreasing series will follow the same patter. In other cases, the increase or decrease will follow a uniform rate rather a constant amount. There are two basic types of gradients series in engineering economic analyses, namely the uniform arithmetic gradient series where the series is a constant amount (positive or negative) and the geometric gradient series which represented by a the uniformly linear rate.

5.1 Uniform Arithmetic Gradient

An arithmetic gradient is a cash flow that either increases or decreases by a constant amount. The cash flow, changes by the same amount each period. The amount of decrease or the increase is the gradient or G. Thus, the arithmetic gradient series cash flow involves **an increase or decrease of a constant amount G in the cash flow of each analysis period**. The receipt or disbursement at a particular time is greater or lower than the receipt or disbursement at the preceding period by a constant amount "G." Suppose that there is a series of "n" payments uniformly spaced but differing from one period to the next by a constant. The change or "gradient" from one period to the next is denoted "G." The G carries a minus sign for decreasing gradients and a positive sign for increasing gradients.

Assume you invested in an electric car battery stock which is expected to pay an amount $A each year. Thus, your first receipt $A and is expected to increase by $G amount each year for the next n years. This means that after the first year, payment is expected to increase by $G. So, the amount you will receive in the second year is $(A+G). The amount in the third year would be $(A+2G), the amount in fourth year $(A+3G), the fifth year ($(A+4G) and so on to the (n-1)th year with the receipt of $\$A + \$G(n-2)$, and finally to the nth year with a receipt of $\$A + \$G(n-1)$. The gradient which (or more specifically the arithmetic gradient) is the equal amount of increase per year which in this case is $G. The cashflow diagram for this scenario is shown in the figure 2.3. It is instructive to note that the gradient G first occurred between year 1 and year 2. One thing to also note is that the base amount of $A is typically not equal to the gradient amount. As earlier indicated, G (the gradient) which is the constant arithmetic change in cash flows from one time period to the next may be positive or negative.

The solution requires breaking the cash flows in to two segments and then combing to give the total equivalent cash flow desired

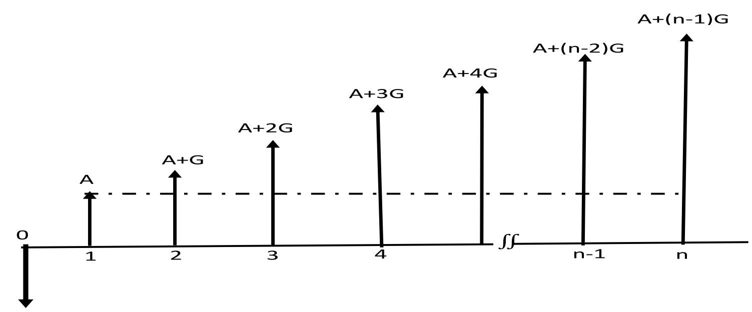

Figure 1. Arithmetic gradient series with the Base Amount

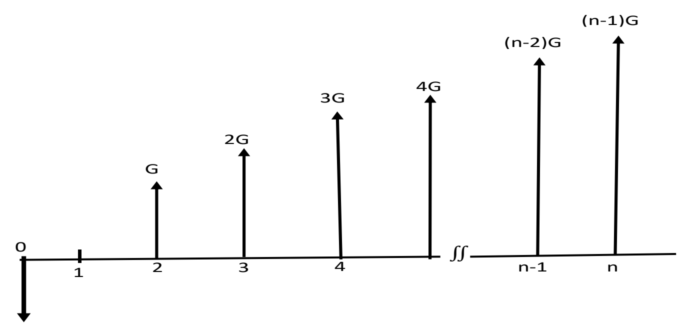

Figure 2. Arithmetic gradient series without the Base Amount

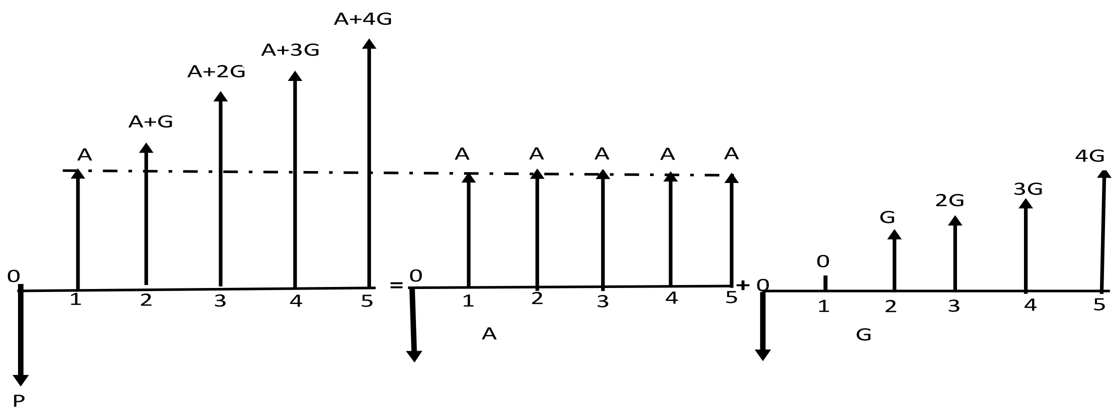

Figure 3. Conversion diagram from the combined cashflow to arithmetic gradient (G) and Annual worth(A)

The present worth at year 0 of only the gradient component is equal to the sum of the present worth of each individual cash flow, where each value is considered a future amount.

This we can represent the present worth of the gradient cashflows as:

$$P = G(P/_F, i, 2) + 2G(P/_F, i, 3) + 3G(P/_F, i, 4) + +[(n-2)G](P/_F, i, n-1) + [(n-1)G](P/_F, i, n)$$

But: $P/_F = \left(\frac{1}{1+i}\right)^n$

$$P = G\left[\frac{1}{(1+i)^2} + \frac{2}{(1+i)^3} + \frac{3}{(1+i)^4} + \frac{4}{(1+i)^5} + \cdots + \frac{(n-3)}{(1+i)^{(n-2)}} + \frac{(n-2)}{(1+i)^{(n-1)}} + \frac{(n-1)}{(1+i)^{(n)}}\right] \quad \ldots\ldots\ldots\ldots\ldots 5.1$$

Multiply P above by (1+i)

$$P(1+\mathrm{i}) = G(1+\mathrm{i})\left[\frac{1}{(1+i)^2} + \frac{2}{(1+i)^3} + \frac{3}{(1+i)^4} + \frac{4}{(1+i)^5} + \cdots + \frac{(n-2)}{(1+i)^{(n-1)}} + \frac{(n-1)}{(1+i)^{(n)}}\right] \quad \ldots\ldots\ldots\ldots\ldots 5.2$$

$$P(1+\mathrm{i}) = G\left[\frac{1}{(1+i)^1} + \frac{2}{(1+i)^2} + \frac{3}{(1+i)^3} + \frac{4}{(1+i)^4} + \cdots + \frac{(n-2)}{(1+i)^{(n-2)}} + \frac{(n-1)}{(1+i)^{(n-1)}} + \frac{n}{(1+i)^n}\right] \ldots\ldots 5.3$$

Subtract 5.1 from 5.3

$$iP = G\left[\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \frac{1}{(1+i)^4} + \cdots + \frac{1}{(1+i)^{n-1}} - \frac{(n-1)}{(1+i)^n}\right]$$

$$iP = G\left[\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \frac{1}{(1+i)^4} + \cdots + \frac{1}{(1+i)^n}\right] - G\left[\frac{n}{(1+i)^n}\right]$$

The Finite series: $\left[\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \frac{1}{(1+i)^4} + \cdots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\right] = \frac{(1+i)^n - 1}{i(1+i)^n}$, How?

Let the sum of the series= $S = \left[\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \frac{1}{(1+i)^4} + \cdots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\right]$

Then: $S(1+i) = \left[1 + \frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \cdots + \frac{1}{(1+i)^{n-2}} + \frac{1}{(1+i)^{n-1}}\right]$

$$S(1+i) - S = \left[1 - \frac{1}{(1+i)^n}\right] \Rightarrow i(S) = \left[1 - \frac{1}{(1+i)^n}\right] = \left[\frac{(1+i)^n - 1}{(1+i)^n}\right] \Rightarrow S = \left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

Therefore, $iP = G\left[\frac{(1+i)^n-1}{i(1+i)^n}\right] - G\left[\frac{n}{(1+i)^n}\right] = G\left[\frac{(1+i)^n-1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right]$

Hence: $P = G\left[\frac{(1+i)^n-1}{i^2(1+i)^n} - \frac{n}{i(1+i)^n}\right] = G\left[\frac{(1+i)^n-in-1}{i^2(1+i)^n}\right]$,

thus: $\boldsymbol{(P/_G, i, n) = G\left[\frac{(1+i)^n-in-1}{i^2(1+i)^n}\right]}$, But: $\{F/_P = (1+i)^n\}$

Hence: $\boldsymbol{(F/_G, i, n)} = (P/_G, i, n)(1+i)^n = G\left[\frac{(1+i)^n - in - 1}{i^2(1+i)^n}\right](1+i)^n = \boldsymbol{G\left[\frac{(1+i)^n - 1 - in}{i^2}\right]}$

Also: $P/_A = \mathrm{A}(\mathrm{P}/_\mathrm{A}, \mathrm{i\%}, \mathrm{n}) = \mathrm{A}\left[\frac{(1+\mathrm{i})^\mathrm{n}-1}{\mathrm{i}(1+\mathrm{i})^\mathrm{n}}\right]$, also $A/_P = \mathrm{P}(\mathrm{A}/_\mathrm{P}, \mathrm{i\%}, \mathrm{n}) = P\left[\frac{i(1+i)^n}{(1+i)^n-1}\right]$

But: $(A/_G, i, n) = (P/_G, i, n)(A/_P, i, n)$

$$(A/_G, i, n) = G\left[\frac{(1+i)^n - 1}{i^2(1+i)^n} - \frac{n}{i(1+i)^n}\right]\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = G\left[\frac{(1+i)^n - 1}{i(1+i)^n - 1} - \frac{(i)n}{i(1+i)^n - 1}\right]$$

Hence: $(A/_G, i, n) = G\left[\frac{1}{i} - \frac{n}{i(1+i)^n - 1}\right]$

The total present worth P_T for a series that includes a base amount A and conventional arithmetic gradient G must consider the present worth of both the uniform series defined by A and the arithmetic gradient series defined by G. The addition of the two results in P_T, that is:
$P_T = P_A + P_G = A(P/_A, i, n) + G(P/_G, i, n)$, where P_A is the present worth of the uniform series only, P_G is the present worth of the gradient series only, and the + or - sign is used for an increasing (+G) or decreasing (-G) gradient, respectively.

The equivalent annual worth $A_T = A_1$ is the sum of the base amount series annual worth A_A and gradient series annual worth A_G, Hence: $A_T = A_A + A_G = A_1 + G(A/_G, i, n)$

The corresponding equivalent future worth is:
$F_T = F_A + F_G = F(F/_A, i, n) + G(F/_G, i, n)$

Note that $\mathrm{A} = \mathrm{F}(\mathrm{A}/_\mathrm{F}, \mathrm{i\%}, \mathrm{n}) = F\left[\frac{i}{(1+i)^n-1}\right]$

Remember: The conventional arithmetic gradient starts in year 2, and P is located in year 0

$$\boxed{\begin{aligned} P_T &= P_A + P_G = A(P/_A, i, n) + G(P/_G, i, n) \\ A_T &= A_A + A_G = A_1 + G(A/_G, i, n) \\ F_T &= F_A + F_G = F(F/_A, i, n) + G(F/_G, i, n) \end{aligned}}$$

5.1.1 Numerical Examples

<u>Problem 5.1</u>. <u>Lawn Mower Example</u>

Assume you purchased the newest riding lawnmower with warranty from Home Depot on a 6-year payment plan. The first payment is \$200 and increases by \$50 each payment due to the warranty. This means that after the first month, payment is expected to increase by \$50 each year. So, the amount to be paid in the second year would be \$250. The amount in the third year would be \$300, the fourth year \$350, and the fifth year \$400 and so on to 6th year(where n=6) with a payment of the total cost would be: $\$200 + 50(n-1) = \$450$. The discount rate is 8% . As shown in figure 4, the problem can be broken down into an equivalent problem with two components, The Base Component (A) and the gradient component (G). Determine the resultant cash flows of F, P, and A.

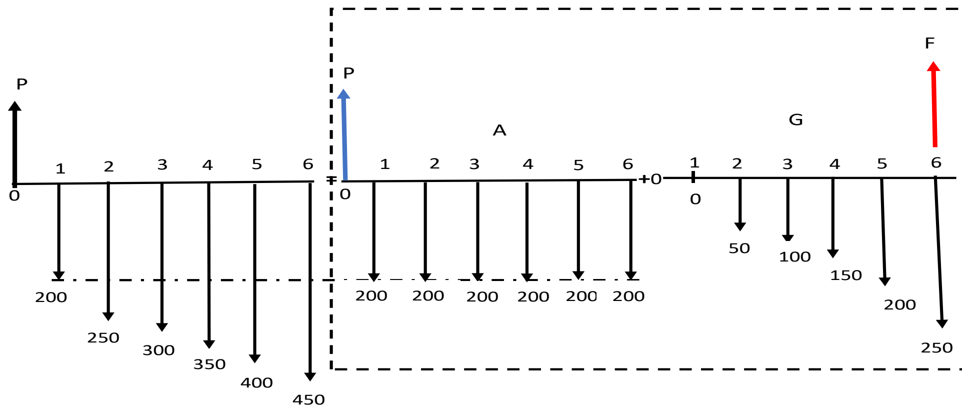

Figure 4. The problem has been separated into two-constituent parts; the Base A and the gradient (G).

The dotted rectangle imposed on figure 4 is used to delineate the new problem. We can now determine the resultant cash flow for the following: Future Value F, Present Value P, and Equivalent Annual Worth A or the Annuity, where:

$$F_T = F_B + F_G = \mathrm{A}\left({}^{F}/_{A}, i\%, n\right) + \mathrm{G}\left({}^{F}/_{G}, i\%, n\right) = \mathrm{A}\left[\frac{(1+i)^n - 1}{i}\right] + \boldsymbol{G}\left[\frac{\boldsymbol{(1+i)^n - 1 - in}}{\boldsymbol{i^2}}\right]$$

$$P_T = P_B + P_G = \mathrm{A}\left({}^{P}/_{A}, i\%, n\right) + \mathrm{G}\left({}^{P}/_{G}, i\%, n\right) = \mathrm{A}\left[\frac{(1+\mathrm{i})^{\mathrm{n}} - 1}{\mathrm{i}(1+\mathrm{i})^{\mathrm{n}}}\right] + \boldsymbol{G}\left[\frac{\boldsymbol{(1+i)^n - in - 1}}{\boldsymbol{i^2(1+i)^n}}\right]$$

$$A_T = A_B + A_G = A + \mathrm{G}\left({}^{A}/_{G}, i\%, n\right) = A + G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right]$$

$$F_T = \mathrm{A}\left[\frac{(1+i)^n - 1}{i}\right] + \boldsymbol{G}\left[\frac{\boldsymbol{(1+i)^n - 1 - in}}{\boldsymbol{i^2}}\right] = 200\left[\frac{(1.08)^6 - 1}{0.08}\right] + \mathbf{50}\left[\frac{\mathbf{(1.08)^6 - 1 - 0.48}}{\mathbf{0.0064}}\right]$$

$$\boldsymbol{\left({}^{F}/_{G}, 0.08, 6\right) = G\left[\frac{(1+i)^n - 1 - in}{i^2}\right] = \left[\frac{(1.08)^6 - 1 - 0.48}{0.0064}\right] = 16.669} \checkmark$$

$$\boldsymbol{F_T = 200(7.3359) + 50(16.669) = 1467.1858 + 834.956 = \$2302.142}$$

$$P_T = \mathrm{A}\left[\frac{(1+\mathrm{i})^{\mathrm{n}}-1}{\mathrm{i}(1+\mathrm{i})^{\mathrm{n}}}\right] + \boldsymbol{G}\left[\frac{(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}-\boldsymbol{in}-\mathbf{1}}{\boldsymbol{i}^{2}(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}}\right] = 200\left[\frac{(1.08)^{6}-1}{0.08(1.08)^{6}}\right] + \mathbf{50}\left[\frac{(\mathbf{1.08})^{\mathbf{6}}-\mathbf{0.48}-\mathbf{1}}{\mathbf{0.0064}(1.08)^{6}}\right]$$

$$\boldsymbol{P_T} = \mathbf{924.5759} + \mathbf{526.1637} = \$\mathbf{1450.7396}$$

$$\boldsymbol{A_T} = \boldsymbol{A} + \boldsymbol{G}\left[\frac{\mathbf{1}}{\boldsymbol{i}} - \frac{\boldsymbol{n}}{(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}-\mathbf{1}}\right] = \mathbf{200} + \mathbf{50}\left[\frac{\mathbf{1}}{\mathbf{0.08}} - \frac{\mathbf{6}}{(\mathbf{1.08})^{\mathbf{6}}-\mathbf{1}}\right] = \mathbf{200} + \mathbf{113.817} = \$\mathbf{313.817}$$

$$\boldsymbol{Thus}: \boldsymbol{F_T} = \$\mathbf{2302.142}, \boldsymbol{P_T} = \$\mathbf{1450.739}, \boldsymbol{A_T} = \$\mathbf{313.817}$$

Please <u>note</u> that once you have computed any of the cash flows, that is, F_T, P_T, A_T, you can obtain the others by multiplying by the appropriate interest factor. For example

$$\boldsymbol{P_T} = \boldsymbol{F_T}\left({}^{\boldsymbol{P}}/_{\boldsymbol{F}}, \boldsymbol{i}\%, \boldsymbol{n}\right) = \boldsymbol{F_T}(1+\mathrm{i})^{-\mathrm{n}} = \$\mathbf{2302.142}(\mathbf{0.630169}) = \$\mathbf{1450.739}$$

$$\boldsymbol{A_T} = \boldsymbol{F_T}\left({}^{\boldsymbol{A}}/_{\boldsymbol{F}}, \boldsymbol{i}\%, \boldsymbol{n}\right) = F_T\left[\frac{i}{(1+i)^{n}-1}\right] = \$\mathbf{2302.142}(\mathbf{0.136315}) = \$\mathbf{313.817}$$

Also note that the table does not provide the gradient factor for future values, F, that is $\mathrm{G}\left({}^{\boldsymbol{F}}/_{\boldsymbol{G}}, \boldsymbol{i}\%, \boldsymbol{n}\right)$ is not given on the gradient table. However, we can get it by using the compound factor relationship among the other cashflows

For example :

$${}^{\boldsymbol{F}}/_{\boldsymbol{G}} = \left({}^{\boldsymbol{A}}/_{\boldsymbol{G}}, \boldsymbol{i}\%, \boldsymbol{n}\right)\left({}^{F}/_{A}, i\%, n\right) = \left[\frac{1}{i} - \frac{n}{(1+i)^{n}-1}\right]\left[\frac{(1+i)^{n}-1}{i}\right] \quad \text{OR}$$

$${}^{\boldsymbol{F}}/_{\boldsymbol{G}} = \left({}^{\boldsymbol{P}}/_{\boldsymbol{G}}, \boldsymbol{i}\%, \boldsymbol{n}\right)\left({}^{F}/_{P}, i\%, n\right) = \left[\frac{(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}-\boldsymbol{in}-\mathbf{1}}{\boldsymbol{i}^{2}(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}}\right]\left[(1+i)^{n}\right]$$

$$\left({}^{F}/_{G}, 0.08{,}6\right) = \left({}^{A}/_{G}, 0.08{,}6\right)\left({}^{F}/_{A}, 0.08{,}6\right) = \left[\frac{1}{i} - \frac{n}{(1+i)^{n}-1}\right]\left[\frac{(1+i)^{n}-1}{i}\right] = 2.2763(7.3359)$$

$$\left({}^{F}/_{G}, 0.08{,}6\right) = 16.669 \checkmark$$

$${}^{\boldsymbol{F}}/_{\boldsymbol{G}} = \left({}^{\boldsymbol{P}}/_{\boldsymbol{G}}, \mathbf{0.08}, \mathbf{6}\right)\left({}^{F}/_{P}, 0.08{,}6\right) = \left[\frac{(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}-\boldsymbol{in}-\mathbf{1}}{\boldsymbol{i}^{2}(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{n}}}\right]\left[(1+i)^{n}\right] = (10.5233)(1.08)^{6}$$

$$\left({}^{F}/_{G}, 0.08{,}6\right) = 16.669 \checkmark$$

<u>Problem 5.1</u> <u>Nitron Oil Well Example</u>

NITRON Global Services has purchased an inactive oil well for \$200m. It is expected tht once activated, the well will produce for another 12 years. It is expected that during the first year of operation, the well will produce an income or cash flow(less operating cost) of \$80m for the first year, \$75m for the second, with annual decrease of \$5m per year. Knowing that the well can only be productive for 12 years, a).what is the equivalent annual yeld, and b). more importantly is this project feasible based on the originl investment \$200m at discount rate of 10% ?

Solution

The future yield of the project: $F_T = F_B + F_G$

$$Base = A = \$80m, G = -\$5m$$

$$F_T = \$80m\left({}^{F}/_{A}, 10\%, 12\right) - \$5m\left({}^{F}/_{G}, 10\%, 12\right)$$

$\$80m\left(F/_{A}, 10\%, 12\right) = \$80m\left[\frac{(1+i)^{n}-1}{i}\right] = \$80m\left[\frac{(1.1)^{12}-1}{0.1}\right] = \$1{,}710.743m$

$-\$5m\left(F/_{G}, 10\%, 12\right) = -\$5m\left[\frac{\boldsymbol{(1+i)^{n}-1-in}}{\boldsymbol{i^{2}}}\right] = -\$469.214\text{m}$

$F_Y = F_T = \$1{,}710.743m - \$469.214 = \$1{,}241.529m$

The Equivalent Annual yield of this cash flow is : $A_Y = F_T\left(A/_{F}, 10\%, 12\right)$

a). $A_Y = F_T\left(A/_{F}, 10\%, 12\right) = \$1{,}241.529m\left[\frac{i}{(1+i)^{n}-1}\right] = \$58.058m$ *per year*

b). The present worth of the yield is:

$P_{Y(t=0)} = A_Y\left(P/_{A}, 10\%, 12\right) = \$58.058m\left[\frac{(1+\mathrm{i})^{\mathrm{n}}-1}{\mathrm{i}(1+\mathrm{i})^{\mathrm{n}}}\right] = \$395.589m$

The Project is feasible if the present value of the yield at time zero is greater than the present value of the investment at time t=0 that is $P_{t=0} < P_{Y(t=0)}$

$P_{Y(t=0)} = \$395.589m$ is greater than $P_{t=0} = \$200m$, Hence the project is feasible

Present Worth of the cash flow **P_T = \$395.589m-\$200m**= **<u>\$195.59m</u>**

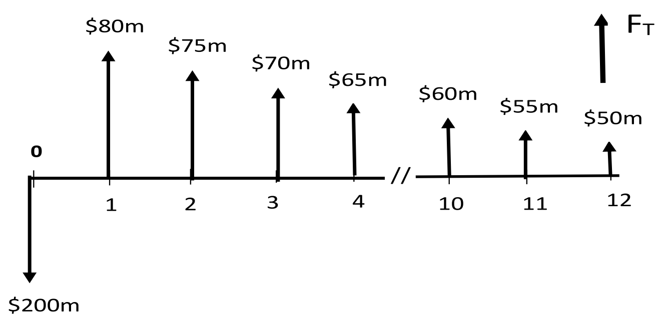

Table 5. The cash flow of the NITRON Oil Yield

<u>OR</u>: $P_T = -P_O + P_A - P_G = \$200m + \$80m\left[P/_{A}, 10\%, 12\right] - \$5m\left[P/_{G}, 10\%, 12\right]$

$P_T = -\$200m + \$80m[6.813] - \$5m[29.901] = (-\$200 + \$545.095 - \$149.505)m = \$\mathbf{195.59}\boldsymbol{m}$

5.2 Shifted Arithmetic Gradient with Base (A=Base)

For conventional gradients, P_O is always located at time t = 0, and the gradient G starts at year 2 or time two. For shifted gradients, G will start at any time other than year 2 or time 2. However, the present worth is still 2 years or two time periods before the gradient commences. As is shown in figure 7 below, we have a cashflow with arithmetic gradient series that is shifted or delayed and does not start until year 7 (or time 7). This means that the original time 5 now becomes the origin or the new time zero where, P_O is located. This means that year 7 where the gradient starts is now the new time 2 which means that year 5 becomes the origin or time 0. Given these adjustments, we can now compute P_T .

To compute the total Present value for arithmetic gradient with base, we proceed as follows:

1. Compute P_G two years before the gradient commences as shown in figure 7. This is now a future value, that is: $P_G = F$
2. Now convert F to P'_G at the origin (t=0) using the factor P/F, where $P'_G = P_G(P/F)$.
3. Find $P_B = P_A$ of the base series at t=0, using the F/A and P/F factors.
4. Finally add these together to get P_T, where $P_T(= P_0) = P_A + P'_G = P_A + P_G(P/F)$

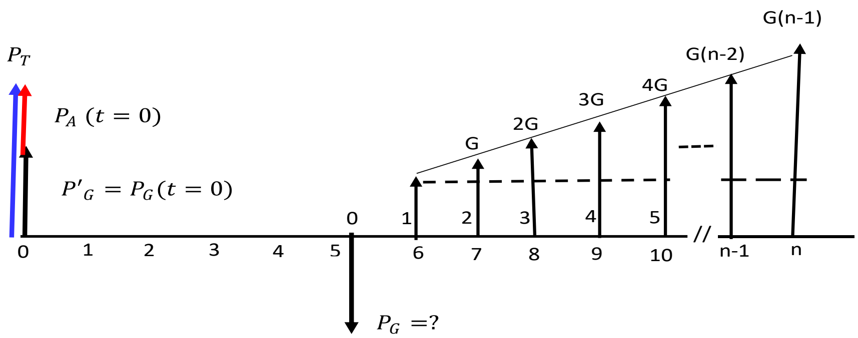

Figure 6. Shifted Arithmetic Gradient

5.2.1 Computation of PW for shifted Arithmetic Gradient

Problem 5.3: Let A=\$5000/year, G=\$500, i=10% annually, n=10 years, shifted by 5 years. Find P_T *or* P_0

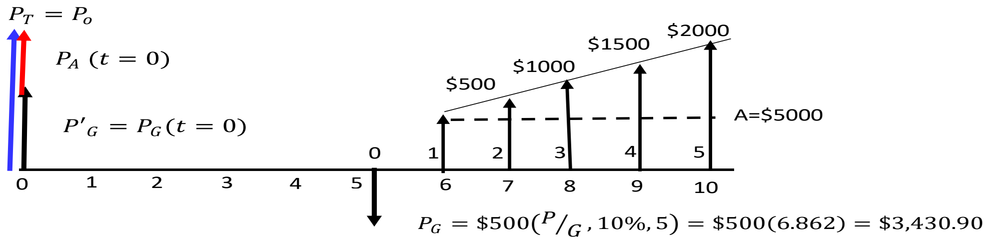

Figure 7: Numerical Example of Shifted Arithmetic Series

$$P_G = G(P/G, 10\%, 5) = \$500\left[\frac{\boldsymbol{(1+i)^n - in - 1}}{\boldsymbol{i^2(1+i)^n}}\right] = \$500(6.682) = \$3,430.9$$

$$P'_G = P_G(P/F, 10\%, 5) = \$3,430.90[(1+\mathrm{i})^{-\mathrm{n}}] = \$3,430.90(0.6209) = \$2,130.245$$

$$P'_A = \$5000(P/A, 10\%, 5) = \$5000\left[\frac{(1+\mathrm{i})^{\mathrm{n}} - 1}{\mathrm{i}(1+\mathrm{i})^{\mathrm{n}}}\right] = \$5000(3.791) = \$18,955.0 = F$$

$$P_A = F(P/F, 10\%, 5) = F(P/F, 10\%, 5) = \$18,955(0.6209) = \$11,769.16$$

OR: $P_A = \$5000(P/A, 10\%, 5)(P/F, 10\%, 5) = \$5000(3.791)(0.6209) = \$11,769.16$
$P_T = P_A + P'_G = \$11,769.16 + \$2,130.25 = \$13,899.41$

5.2.2 Computation of AW for shifted Arithmetic Gradient

Problem 5.4: Let A=\$5000/year, G=\$500, i=10% annually, n=1-10 years.

$A_T = A_A + A_G$

$$A_G = P'_G(A/_P, 10\%, 10) = \$2{,}130.25\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = \$2{,}130.25(0.1627) = \$346.59$$

$$A_B = P_A(A/_P, 10\%, 10) = \$11{,}769.16\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = \$11{,}769.16(0.1627) = \$1{,}914.84$$

Thus, $\boldsymbol{A_T = A_A + A_G = \$346.59 + \$1{,}914.84 = \$2{,}261.43/year}$

Alternatively, $\boldsymbol{A_T = P_T(A/_P, 10\%, 10) = \$13{,}899.41(0.1627) = \$2{,}261.43/year}$

Thus, $\boldsymbol{F_T = F_A + F_G = P_A(F/_P, 10\%, 10) + P'_G(F/_P, 10\%, 10) = P_A(1+i)^n + P'_G(1+i)^n}$
$\boldsymbol{F_T = \$11{,}769.16(2.594) + \$2130.25(2.594) = \$36{,}055.10}$

Alternatively, $\boldsymbol{F_T = P_T(F/_P, 10\%, 10) = \$13{,}899.41(2.594) = \$36{,}055.10}$

5.2.3 Numerical Example of Shifted Arithmetic Gradient

Problem 5.5. The BIDAN company is considering investing \$100,000 in a project that has a life of 6 years. Due to the nature of the project, the Net Cash Flow (NCF) is delayed till end of year 2 with an income of \$40,000 and increased by \$40,000/year until the end life of the project, that is, project income is as follows (assume i=8% annually):

Year 0 = -\$100K, Year 1 = \$0K , Year 2 = \$0K

Year 3 = \$40K , Year 4 = \$80K, Year 5 = \$120K, Year 6 = \$160K

As discussed earlier, we know that for conventional gradients, P_O is always located at time t = 0, and the gradient G starts at year 2 or time two. For shifted gradients, G could start at any time other than year 2 or time 2. However, the present worth is still 2 years or two time periods before the gradient commences. From the problem description and figure 8, we can see that this is a shifted Arithmetic gradient, and that the gradient begins in year 3 rather than the usual year 2. So, this means we will adjust the numbering to reflect the shift. The numbering above the line is the new numbering while the one below is the original numbering

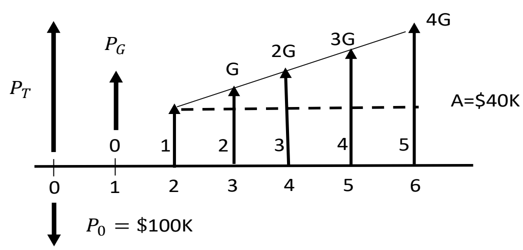

Figure 8. Shifted Arithmetic gradient

$$P_G = G\left(\frac{P}{G}, 8\%, 5\right) = G\left[\frac{\mathbf{(1+i)^n - in - 1}}{\mathbf{i^2(1+i)^n}}\right] = \$\mathbf{40,000(7.372)} = \$\mathbf{294,880}$$

$$P_T = -P_0 + P_G\left(\frac{P}{F}, 8\%, 5\right) = -\$100{,}000 - \$294{,}880(0.6806) = +\$\mathbf{100,6990.37}$$

5.3 Geometric Gradient

A geometric gradient series is **a cash flow series that either increases or decreases by a constant percentage each period**. The uniform change also called the rate of change in decimal form, where: g , is a constant rate of change (or the growth rate (+ or -) by which cash flow values increase or decrease from one period to the next. In the case of the Arithmetic gradients, the cash flows change by a constant amount from one interest period to the next. However, for Geometric gradients, the cash flows change by a constant percentage equal to g.

Suppose that there is a series of "n" payments uniformly spaced, but differing from one period to the next by a constant multiple. The change or "gradient" multiple from one period to the next is denoted "g." There will, of course, also be an interest rate "i" that applies. Define the following

$A_1 = Value\ of\ initial\ cash\ Flow\ (Typically\ Year\ 1)$

$A_t = Value\ of\ cash\ Flow\ at\ any\ year\ t$

$g = uniform\ rate\ of\ cash\ flow\ increase\ (decrease)\ from\ one\ interest\ period\ to\ another$

Note that P (the present Worth) is located at period 0 and the first change in the gradient g occurs in period 2, just like the arithmetic gradient. P represents the Present worth of all the cash flows between period 1 and n.

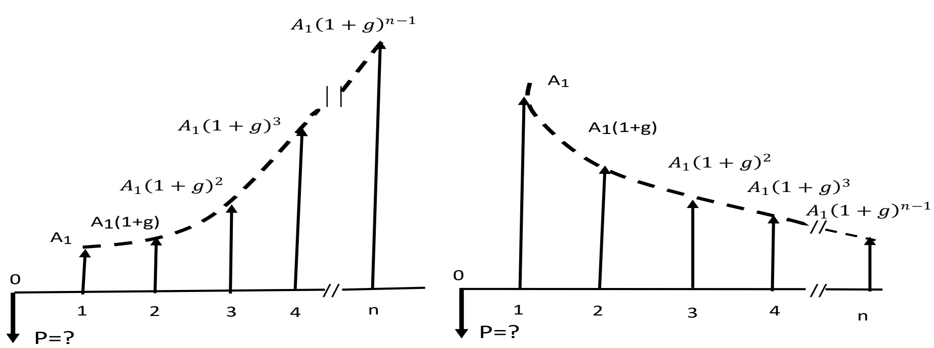

Figure 9: Geometric Series

In general: $A_t = A_1(1+g)^{t-1}$

Also, as discussed earlier when we developed the relationship between the present value P and any cash flow (F or A, where the A in this case is not necessarily the annuity) with interest rate i, we have:

$F_t = P_t(1+i)^t$

$\Rightarrow P_t = F_t(1+i)^{-t}$
But: $A_t \; = Value\ of\ cash\ Flow\ at\ any\ year\ t$
$\Rightarrow P_t = A_t(1+i)^{-t}$
Hence :
$P = A_t(1+i)^{-t}$, But $A_t = A_1(1+g)^{t-1}$

$Hence,\ \ P = A_1(1+g)^{t-1}(1+i)^{-t}$…………7.4
Multiply the RHS of equation 7.4 by the quantity $\left[\frac{(1+i)^{-1}}{(1+i)^{-1}}\right]$

$$P = A_1(1+g)^{t-1}(1+i)^{-t}\left[\frac{(1+i)^{-1}}{(1+i)^{-1}}\right] \Rightarrow A_1(1+g)^{t-1}(1+i)^{-1}\left[\frac{(1+i)^{-t}}{(1+i)^{-1}}\right]$$

Hence: $P = A_1(1+i)^{-1}\left[\frac{(1+g)^{t-1}}{(1+i)^{t-1}}\right] = A_1(1+i)^{-1}\left(\frac{1+g}{1+i}\right)^{t-1}$

In general, it can be shown that

$$P = A_1(1+i)^{-1}\left(\frac{1+g}{1+i}\right)^{t-1} = A_1(1+i)^{-1}\sum_0^n\left(\frac{1+g}{1+i}\right)^{t-1} \text{……..7.5}$$

Expanding and solving, the finite series, we finally have

$P = A_1\left[\frac{1-(1+g)^n(1+i)^{-n}}{i-g}\right]$, where i ≠ g
Hence: $(P/A, g, i, n)) = \left[\frac{1-(1+g)^n(1+i)^{-n}}{(i-g)}\right]$,
If i =g
then $P = A_1[n(1+i)^{-1}] \Rightarrow P = A_1\left[\frac{n}{(1+i)}\right]$, and
$(P/A, g, i, n)) = [n(1+i)^{-1}] = \left[\frac{n}{(1+i)}\right]$
Please note: We can also express the Geometric gradient in terms of future values F.
We know: $F = P(1+i)^n$
But $P = A_1\left[\frac{1-(1+g)^n(1+i)^{-n}}{(i-g)}\right]$
Hence: $F = P(1+i)^n \Rightarrow A_1\left[\frac{1-(1+g)^n(1+i)^{-n}}{(i-g)}\right](1+i)^n = A_1\left[\frac{(1+i)^n-(1+g)^n(1+i)^{-n}(1+i)^n}{(i-g)}\right]$

$F = A_1\left[\frac{(1+i)^n-(1+g)^n(1+i)^{-n}(1+i)^n}{(i-g)}\right] = A_1\left[\frac{(1+i)^n-(1+g)^n}{(i-g)}\right]$, where i ≠ g
If i =g
Recall that when i=g, $P = A_1[n(1+i)^{-1}]$=$A_1\left[\frac{n}{(1+i)}\right]$
But $F = P(1+i)^n$, hence $F = A_1\left[\frac{n}{(1+i)}\right](1+i)^n \Rightarrow A_1 n(1+i)^{n-1}$

Example 5.6

Orange County commissioners want to know the present worth of the maintenance cost of its new solar energy monitoring software over the next 6 years. The maintenance cost at the end of the first year (.i.e. A_1) is $10,000 and the cost is expected to increase by 5 % each year and the interest rate of 8%.

a). What is the present worth of the maintenance cost.
b). Alternatively find the future worth of the maintenance costs.
c). Show that the answer to part (b) is correct based on the answer to part (a).

Solution: The cash flow by the end of the first year is $10,000, which increases by 5% each year.
$A_1 = 10{,}000, g = 5\%, i = 8\%, n = 6\ years$
a).

$$P = A_1\left[\frac{1-(1+g)^n(1+i)^{-n}}{i-g}\right] = \$10{,}000\left[\frac{1-\frac{(1.05)^6}{(1.08)^6}}{0.08-0.05}\right] \Rightarrow \$10{,}000[5.18375] = \$51{,}377.477$$

Hence the present worth is $P = \$51{,}837.477$

b). $F = A_1\left[\frac{(1+i)^n-(1+g)^n}{(i-g)}\right] = \$10{,}000\left[\frac{(1.08)^6-(1.05)^6}{0.03}\right] = \$10{,}000\left[\frac{0.247}{0.03}\right]$

Hence the present worth is $F = \$82{,}259.56$

c). $F = P(1+i)^n = \$15{,}837.477(1.08)^6 = \$51{,}837.476(1.08)^6 = \$82{,}259.56$, OR

$P = F(1+i)^{-n} \Rightarrow F\left[\frac{1}{(1+i)^n}\right] = \$82{,}259.56\left(\frac{1}{(1.08)^6}\right) = \$51{,}837.477$

Example 5.7

NITRON Global Services is in the process of approving a 4% per year raise for its four senior management staff over the next 5 years. If the salary of each of the five-executive staff at the end of this year is $200,000, what is the present worth of the combined salary of the five designated staff, assuming an interest rate of 10% .

Solution

$A_1 = 200{,}000(5) = \$1m, g = 4\%, i = 10\%, n = 5\ years$

$$P = A_1\left[\frac{1-(1+g)^n(1+i)^{-n}}{i-g}\right] = \$1m\left[\frac{1-\frac{(1.04)^5}{(1.1)^5}}{0.10-0.04}\right] = \$4.076m = \$4{,}076{,}000.00$$

5.4 Shifted Geometric Gradient

As previously shown for conventional gradients, P_O is always located at time t = 0, and the geometric gradient (g) starts at year 2 or time two after the gradient commences. For shifted gradients, the gradient g will start at any time other than year 2 (or time 2) and the present worth of the geometric gradient P_g is located 2 periods before the gradient starts. P_g includes the initial amount A_1 and again, no base series is considered separately as is the case with arithmetic gradient.

Consider the figure 8 below. The geometric series starts in actual year 4. P_g is located in actual year 3 (or adjusted year 0 due to the shift) before the gradient starts. Initial amount of the gradient A_1 starts in actual year 4 (or adjusted year 1 due to the shift). There is a series **A** equal in size to A_1 that goes from year 1 to year 9. P_0 the initial investment is located in year 0. The growth rate is g, and the interest rate is i. The goal is to find total present worth P_T for the cash flows.

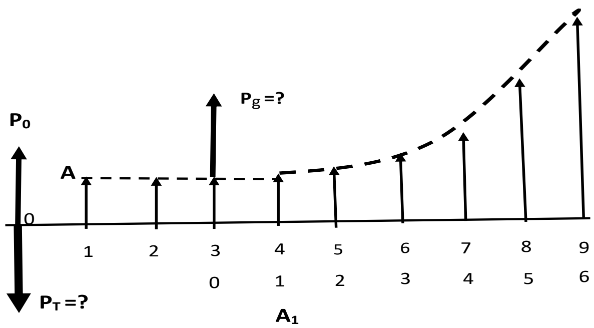

Figure 10: Shifted Geometric Series.

Problem 5.8: Seminole County has signed a 4-year maintenance contract with a software developer for an enterprise software that would monitor the solar energy usage in the county as part of its Green Economy initiative. The initial cost of the software is $300,000. The maintenance cost is $15,000 per year for the next 4 years. At the end of the contract in 4 years, the maintenance cost will increase by 10% per year for the next 5 years after which the county would decide to continue with the software or bring a different system on board. What is the present worth P_T of the cash flows if the interest rate is 15%?

Please note that as with any gradient, the geometric gradient starts in year 1 with $A=A_1$ (actual year 4 in our example) although the growth starts in year 2 (actual year 5 in this case). The computation P_g starts at year 1 which makes $n_g = 6$.

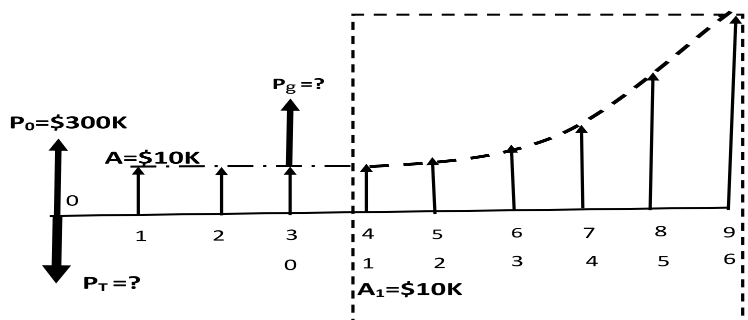

Figure 11: Shifted Geometric Series Numerical Example

$$P_0 = \$300K, A_1 = \$10K, g = 10\%, i = 15\%, n_g = 6$$

$$P_g = \$10{,}000(P/A, i\%, g\%, n_g) = \$10{,}000(P/A, 15\%, 10\%, 6) = \$10{,}000\left[\frac{1-\left[\frac{(1.10)^6}{(1.15)^6}\right]}{(0.15-0.10)}\right]$$

$$P_g = \$10{,}000(4.6821) = \$4{,}682.11$$

$P_{A0} = A(P/A, 15\%, 3) = \$10{,}000(2.2832) = \$22{,}832.00$

$P_T = P_0 + P_{A0} + P_g(P/F, 15\%, 3)$
Note that P_g is a future value (at t=3) so, it must be taken to time t=0 to for its present value.
$Thus, P_g(P/F, 15\%, 3) = \$4{,}682.11(0.6575) = \$3{,}078.56$
Hence: $P_T = \$300{,}000 + \$22{,}832.00 + \$3{,}078.56 = \$325{,}910.56$

Cash Flow Analysis

Cash flow represent the net amount of cash and cash equivalents that flows in and out of an organization. Cash received represents inflows, while cash spent represents outflows. The cash flow statement is a financial statement that shows a company's sources and usage of cash over certain periods. A company's cash flow is typically segmented into flows from operations, investments, and borrowed funds. An organization's ability to creative value and economic sustainability is contingent on its ability to generate positive cash flows and more specifically, the ability to optimize long-term free cash flow (FCF). FCF is the cash generated by a company from its normal business operations less cash expended on capital expenditures (CapEx).

In general, CapEx are funds used by an organization to acquire, upgrade, and maintain physical assets such as infrastructure, plants, buildings, technology, software, and equipment. It is used to execute new projects or investments. Capital expenditures can also be made to improve fixed assets such as roof repair, replacing an old equipment or purchasing a new one, and/or the construction of new facilities. For the most part this type of investment is made to increase the scope of operations or to add economic value to a company's operation

6.1 Future Value (F/ FV)

The Future value (F) is a concept that is derived from the all-important notion of time value of money (TVM). It represents the value of a series of cash flows at a point in time in the future. The value of F is derived by calculating how much the present value (P) of an asset or cash would be worth at a specific time in the future. For example, you would be calculating the future value (F) if you were interested to know how much the \$20,000 you planned to set aside today for a college fund would be worth in 18 years. From another investment perspective, one would be looking to find the future value of a retirement account if a contribution of \$1,000 was made monthly for the next 15 years.

From the point of view of future value, ten dollars put into a savings account today might be worth more than ten dollars a year from now because the bank pays interest on the money in the savings account for that year. Thus, a dollar deposited today has a higher future value due to the time value of money. This is also true for investments in projects. Thus, the notion of F (future value) is an important consideration because it tells investors and individuals how much an investment made today (present value) will be worth in the future.

In general, $F = P(1+i)^n$, and hence $P = \frac{F}{(1+i)^n}$

For daily confounding over n years, $F = P\left(1 + \frac{i}{360}\right)^{360n}$

For continuous confounding, $F = Pe^{in}$

6.2 Present Value (P)/Present Worth (PW)

Present value (PV or P) is the current value of a future sum of money or stream of cash flows realized from a specified rate of return that is expected in the future. This value will differ from the cash flows' nominal value since time itself affects the value of future cash flows. Cash flows are discounted at the discount rate, and the higher the discount rate, the lower the present value of the future cash flows. Determining the appropriate discount rate is the key to properly valuing future cash flows, whether they be earnings or debt obligations. The present value of a project or an outlay determines what the cash flow to be received in the future is worth in today's dollars. Present value discounts the future cash flow back to the present date, using the average rate of return or the WACC and the number of periods of the life of the project. All things being equal, if a present value amount is invested at a specified rate of return (discount rate or WACC) for a number of periods, the investment would grow into the future cash flow amount.

Present Value: $\boldsymbol{P = \frac{F}{(1+r)^n}}$,

Where: ***F=Future Value, r =Rate of return (%), n = Number of periods***

6.3 Annual Cost (A)/ Annual Cost (AW)

In many instances we encounter uniform series of payments or receipts such as mortgage payment or payments for automobile loans. Akin to this is the "equivalent annual cost" which refers to the cost-per-year of owning, operating, and maintaining an asset over the course of its entire lifespan. Firms often use EAC for capital budgeting decisions, as it allows a company to compare the cost-effectiveness of various assets with unequal lifespans based on their annual costs. The uniform cash flow or outlay is an annual cash flow denoted by "A" for all period lengths of one year.

Suppose we have Equal Annual Payment "A" for 5 years, that is equal payments per period, where each period is one year, and payments take place at the end of each year.

We know that those future payments can be expressed as: $F = P(1+i)^n \Rightarrow P = \frac{F}{(1+i)^n}$

We know that those F values are Equivalent Annual payments, so $P = \frac{A}{(1+i)^n}$

For year 1: $P_1 = \frac{A}{(1+i)^1}$, year 2: $P_2 = \frac{A}{(1+i)^2}$, year 3: $P_3 = \frac{A}{(1+i)^3}$, year 4: $P_4 = \frac{A}{(1+i)^4}$, year 5: $P_5 = \frac{A}{(1+i)^5}$

The Present Value P of all the equivalent annual payments is

$P = \frac{A}{(1+i)^1} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \frac{A}{(1+i)^4} + \frac{A}{(1+i)^5}$ …………6.1

For n periods,

$$P = \frac{A}{(1+i)^1} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \frac{A}{(1+i)^4} + \frac{A}{(1+i)^5} \ldots \ldots \ldots . \frac{A}{(1+i)^{n-1}} + \frac{A}{(1+i)^n}$$

Multiplying both sides of equation 4.1 by (1+i)

$$P(1+i) = A + \frac{A}{(1+i)^1} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \frac{A}{(1+i)^4} + \frac{A}{(1+i)^5} \ldots\ldots \frac{A}{(1+i)^{n-1}} \ldots\ldots\ldots 6.2$$

Subtract 6.1 from 6.2: $P(1+i) - P = A - \frac{A}{(1+i)^n}$

$$P[1+i-1] \Rightarrow P(i) = A\left[1 - \frac{A}{(1+i)^n}\right] \Rightarrow A\left[\frac{(1+i)^n - 1}{(1+i)^n}\right] = \frac{A[(1+i)^n - 1]}{(1+i)^n}$$

Simplifying:

$P(i) = \frac{A[(1+i)^n-1]}{(1+i)^n} \Rightarrow P = \frac{A[(1+i)^n-1]}{i(1+i)^n}$, thus: $P = A\left[\frac{(1+i)^n-1}{i(1+i)^n}\right]$, **and hence:** $A = P\left[\frac{i(1+i)^n}{(1+i)^n-1}\right]$

$But: P = \frac{F}{(1+i)^n} \Rightarrow P = F(1+i)^{-n}$, and $hence: \ F = P(1+i)^n$

Since: $F = P(1+i)^n,\ then: F = \left[\frac{A[(1+i)^n-1]}{i(1+i)^n}\right](1+i)^n \Rightarrow F = A\left[\frac{(1+i)^n-1}{i}\right]$, **hence:** $\boldsymbol{A} = \boldsymbol{F}\left[\frac{i}{(1+i)^n-1}\right]$

For n=∞,$P = \frac{A[(1+i)^\infty-1]}{i(1+i)^\infty} = \frac{A[\infty-1]}{i\infty} = \frac{A(\infty)}{i(\infty)} = P = \frac{A}{i} = \frac{AW}{i} =$ for perpetuity

6.4 Summary of cash Flows and Interest Formulas

Future Cash Flow (F), given Present Cash Flow (PW or P): $\mathrm{F} = \mathrm{P}(\mathrm{F}/_\mathrm{P}, \mathrm{i}\%, \mathrm{n}) = \mathrm{P}[(1+i)^n]$

Present Worth Cash Flow (or P) given Future Cash Flow (F): $\mathrm{P} = \mathrm{F}(\mathrm{P}/_\mathrm{F}, \mathrm{i}\%, \mathrm{n}) = \mathrm{F}[(1+\mathrm{i})^{-\mathrm{n}}]$

Future Cash Flow (F), given Annual Cash Flow (PW or P): $F = \mathrm{A}(F/_A, i\%, n) = \mathrm{A}\left[\frac{(1+i)^n-1}{i}\right]$

Annual Cash Flow(A), given Future Cash Flow (F): $\mathrm{A} = \mathrm{F}(\mathrm{A}/_\mathrm{F}, \mathrm{i}\%, \mathrm{n}) = F\left[\frac{i}{(1+i)^n-1}\right]$

Present Worth (PW or P), given Annual Cash Flow(A): $P = \mathrm{A}(\mathrm{P}/_\mathrm{A}, \mathrm{i}\%, \mathrm{n}) = \mathrm{A}\left[\frac{(1+\mathrm{i})^\mathrm{n}-1}{\mathrm{i}(1+\mathrm{i})^\mathrm{n}}\right]$

Annual Cash Flow(A), given Present Cash Flow (P): $\mathrm{A} = \mathrm{P}(\mathrm{A}/_\mathrm{P}, \mathrm{i}\%, \mathrm{n}) = P\left[\frac{i(1+i)^n}{(1+i)^n-1}\right]$

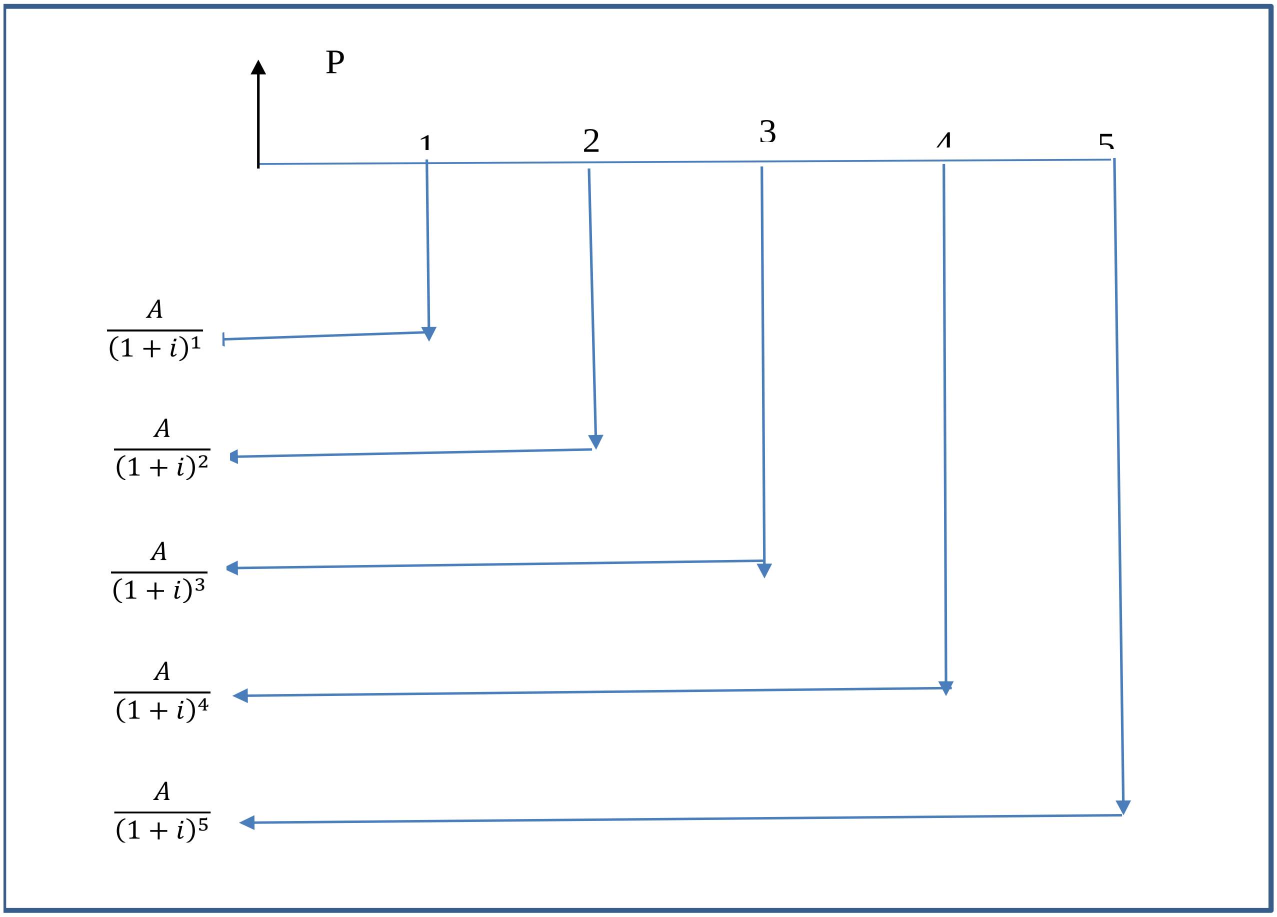

Figure 12. Functional relationship between the Present value (P) and Annual Equivalent (A)

Capital Budgeting

7.1 Discount Rate--DR

The discount rate is the interest rate charged to commercial banks and other financial institutions for short-term loans they take from the Federal Reserve Bank. For our purposes, the discount rate refers to the interest rate used in discounted cash flow (DCF) analysis to determine the present value of future cash flows. In discounted cash flow analysis, the discount rate expresses the time value of money and can make the difference between whether an investment project is financially viable or not.

7.2 Discounted and Non-Discounted Cash Flow

Discounted Cash Flow

Discounted cash flow(DCF) is a method of evaluation for estimating the value of an investment based on the investment expected future cash flows which is dependent on the time value of money

(TVM). It attempts to determine the value of an investment today based on the cash flows generated by the investment in the future. It helps in making capital budgeting or operating expenditures decisions

Discounted cash flow is a tool used in capital budgeting for investment valuation. It is idea of calculating the value of its future cash flow projections based on the concept Time Value of money (TVM) and that is that money in the future is not worth as much as the same amount today.

. The concept of Discounted Cash flow (DCF) derives from the fact that in order to calculate the future worth of any amount of money, it has to be discounted due to the time value of money or account for the lost opportunity to invest and earn interest from it. It is the Opportunity Cost of capital (also referred to as cost of capital) or the return on an investment that an organization would have earned if it did not invest in the current project. In other words, it is the rate of return that an organization is willing to forgo to invest on a preferred project. For example, a company has an opportunity to invest in two projects, a new CNC machining center with a return of 12% and a new software with a return of 15%. Assume the two projects are mutually exclusive. If the company decides to invest in the CNC machine, then the opportunity cost to the company is 15% because it lost the opportunity to invest in the software. If on the other had it decided to buy the new software, then the opportunity cost is lost opportunity to invest in the CNC machine of 12%.

DCF is the sum of all future cash flows of a given project but discounted to the present since money in the future is worth less than it is today.

Non-discounted Cash Flows

Non-discounted method of capital budgeting is one whose cash flows do not incorporate the time value of money (TVM) and solely consider the current value of cash flows when it comes to making investment decisions. In other words, all dollars earned in the future are assumed to have the same value as today's dollars. Since, non-discounted cash flows do not consider the reduction in the value of money over time, it does not support realistic or accurate investment decisions because it tends to overstate the Net Present Value (NPV). On the other hand, discounted cash flows are cash flows that take into consideration the time value of money

One example of a non-discount method is the payback method, since (as we will find out later) does not consider the time value of money. The payback method simply computes the number of years it will take for an investment to return cash that is equal to the amount invested. The resulting number of years is referred to as the payback period. The payback method is only interested in how long before the cash invested is returned. and does not address which of the investments is more profitable? The payback method does not address which investment is more profitable. Note from our examples that the payback method not only ignores the time value of money, it ignores all of the cash received after the payback period.

As an example of payback period, suppose an investor invests $200,000 today in a project with the expectation that generate cash inflow $40,000 for four years followed by $30,000 per year for two additional years, and $20,000 in years seven through eight. Then the payback period is five years and 4 months.

($40,000 + $40,000+ $40,000+ $40,000 +$30,000+$10,000).

Note that since the cash inflow in years five and six is equal to $2500 each month, then the number of months to accumulate $10,000 is 4 months($2500/month multiplied by 4=$10,000).

Another example of a non-discount method in capital budgeting is return on investment (ROI). In ROI, the accounting amounts (revenues, expenses, asset book values, etc.) ignore the time value of money. ROI is the ratio of the net return to the cost of return multiplied by 100. That is:

$$ROI = \left(\frac{Net\ Return}{Cost\ of\ Investment}\right) * 100 .$$

Stated differently it is the current value of investment less the cost of investment all divided by the cost multiplied by 100, that is: $ROI = \left(\frac{Current\ value\ of\ Investment - Cost\ of\ Investment}{Cost\ of\ Investment}\right) * 100$

From these examples it is clear that the payback method not only ignores the time value of money but also all the cash received after the payback period.

7.3 Net Present Value (NPV)

Net present value (NPV) is the difference between the present value of cash inflows because of an investment or project and the present value of the initial cost or cash outflows from the investments over a period. As such, it is used to analyze the profitability of a proposed investment or project and represents the current total value of a future stream of payments or income. If the NPV of a project or investment is positive, then it means that the discounted present value of all future cash flows related to such a project or investment would be positive, and hence the NPV for such a project or investment is considered as acceptable or worthwhile. In order words, any potential project or investment with a negative NPV should not be considered.

The whole notion of NPV is anchored around the idea of the importance of time value of money (TVM). This makes it a particularly valuable tool in comparing similar investment alternatives. NPV assesses the profitability of an investment based on TVM, namely, the idea that an amount of money in the future is not worth the same amount today because money loses its value over time due to inflation. However, money invested today can earn a return, making its future value higher than the same amount received at some point in time in the future. NPV seeks to determine the present value of an investment's future cash flows over and above the investment's initial cost. The discount rate element of the NPV formula is derived from the cost of the capital required to make the investment and is used to discounts the future cash flows to the present-day value.

Subtracting the initial cost of the investment from the sum of the cash flows in the present day would result in a remainder referred to as NPV. If the quantity NPV is positive, then the investment is considered worthwhile. The NPV dictum is that only projects with positive NPV are to be given further consideration. Define:

- PV= Present Value of Invested Cash=Initial Cash Investment
- PVECF= Present Value of Expected Cash Flow
- R_t=Net cash inflow-outflows during a single period t
- i=Discount rate or return that could otherwise be earned in alternative investments
- t=Number of time periods

Let: $PVECF = \sum_{t=1}^{n} \frac{R_t}{(1+i)^t}$

The NPV formula is given as: $\boldsymbol{NPV} = -\boldsymbol{PV} + PVECF \quad = -\boldsymbol{PV} + \sum_{t=1}^{n} \frac{R_t}{(1+i)^t}$

$$\begin{Bmatrix} If\ NPV \quad > 0\,, & Then\ Investment\ is\ \ acceptable/worthwhile \\ If\ NPV \quad < 0, & Then\ Investment\ is\ \ NOT\ acceptable/Not\ worthwhile \end{Bmatrix}$$

7.3.1 Positive vs. Negative NPV

A positive NPV indicates that the projected earnings generated by a project or investment in today's dollars exceeds the anticipated costs, also in today's dollars. Hence an investment with a positive NPV will be profitable. Alternatively, an investment with a negative NPV will result in a net loss.

For example, an investor could receive \$5,000 today or a year from now. Most investors would rather receive \$5,000 today than a year from now. However, if the investor has the option of \$5,000 today or \$5,500 at the end of the year, then everything else being equal, the 10% rate of return (RoR) on this investment for one year seems reasonable unless of course there is another investment with an RoR higher that 10% which implies the receipt from such an investment would be numerically higher than \$5,500, where the RoR in this example is computed as follow:

$$RoR = [\$(5500 - 5000) \div \$(5000)]x100 = 10\%$$

If the investor is assured that another relatively safe investment has an RoR of 12% say over the next year, then a rational investor (everything else being equal) would not be willing to postpone payment of 12% which is the investor's discount rate in this scenario. In this scenario, the yield of the investment after one year at an RoR of 12% is computed as follows: Let the value of the investment after one year be denoted by X, then $\left(\frac{X-5000}{5000}\right)(100) = 12\%$,

$$[\$(x - 5000) \div 5000] = 0.12 \Rightarrow\Rightarrow X == \$\big(5000 + 0.12(5000)\big) = \$5,000 + \$6,000 = \$5,600$$

7.3.2 Computation of the Net Present Value (NPV)

Quite often, in an attempt to determine an appropriate rate of return (RoR) for a project or an investment, an organization may decide to use the expected return of other projects with the same risk level as the yardstick. Alternatively, it may decide to base its expected rate of return on the lending rate from the bank which represents the cost of borrowing the money for the investment. Typically, companies do not get involved in a project if the cost to finance the project (based on the discount or lending rate) is more than the expected rate of return. For example, if it cost 10% to finance a project while the RoR is 7%, a company would be hard pressed to embark on such a project. If on the other hand an alternative project would return 15% while the cost of capital is 10%, then every other thing being equal, the company would most likely make the investment because it makes economic sense.

Money in the present is worth more than the same amount in the future due to inflation and possible earnings from alternative investments that could be made during the intervening time. In other words, a dollar earned in the future won't be worth as much as one earned in the present. The discount rate element of the NPV formula is a way to account for this.

For example, assume that an investor could choose a $100 payment today or in a year. A rational investor would not be willing to postpone payment. However, what if an investor could choose to receive $100 today or $105 in a year? If the payer was reliable, that extra 5% may be worth the wait, but only if there wasn't anything else the investors could do with the $100 that would earn more than 5%.

An investor might be willing to wait a year to earn an extra 5%, but that may not be acceptable for all investors. In this case, the 5% is the discount rate, which will vary depending on the investor. If an investor knew they could earn 8% from a relatively safe investment over the next year, they would not be willing to postpone payment for 5%. In this case, the investor's discount rate is 8%.
A company may determine the discount rate using the expected return of other projects with a similar level of risk or the cost of borrowing the money needed to finance the project. For example, a company may avoid a project that is expected to return 10% per year if it costs 12% to finance the project or an alternative project is expected to return 14% per year.

7.3.3 Examples

CASE I: Suppose a company is amenable to spending $4.5million to purchase either of two types of equipment or alternatively invest the money in the stock market. For one of the equipment, there is a projected $85,000 monthly inflow for 8 years or 96 months. At the end of the eight years, the equipment or project has a salvage value of zero.

CASE II: The company also has the option of purchasing another equipment that has a shorter useful life that would generate a revenue of $100,000 monthly for 6 years or 72 periods/months with a salvage value of zero.

Rate of Return/Discount rate: The company has the capital on hand and could alternatively invest it in the stock market for an expected return of 9% per year. The managers feel that buying the equipment or investing in the stock market have similar risk profile.

Effective Annual rate $i_a = \left[\left(1+\frac{r}{m}\right)^m - 1\right] = \left[\left(1+\frac{r}{12}\right)^{12} - 1\right]$

Monthly Periodic rate: $\frac{r}{m} = \left[(1+i_a)^{\frac{1}{m}} - 1\right] = \left[(1+i_a)^{\frac{1}{12}} - 1\right]$

7.3.4 Steps for Computing the Net Present Value (NPV)

The NPV can be evaluated based on two key metrics, namely the initial investment and the future cash flows. The following example demonstrates how to compute NPV.

Step 1: NPV of the initial investment

CASE I: Consider the of the initial investment of $4,500,000. Monthly or periodic outflow is $85,000 for 8 years or 96 months/periods. In this scenario, because the equipment is paid for upfront, this is the first cash flow included in the calculation. No elapsed time needs to be accounted for, so today's outflow of $4,500,000.00 does not need to be discounted, the two important parameters in this case are the number of periods (t) and the discount rate (r).

- **The Number of periods (t):** The equipment is expected to generate monthly cash flow and last for eight years, which means there will be ninety-six cash flows and ninety-six periods included in the calculation. t = **8x12=96 periods**
- **Discount rate (i):** The alternative investment is expected to pay 9% per year. However, because the equipment generates a monthly stream of cash flows, the annual discount rate must be turned into a periodic or monthly rate. Using the following formula, the effective periodic rate is

Effective Annual rate $=\left[\left(1+\frac{0.090}{12}\right)^{12}-1\right]=0.093=9.3\%$

Monthly discount rate $=\left[(1+0.093)^{\frac{1}{12}}-1\right]=0.0074\ or\ 0.74\%$

<u>CASE II</u>: Consider the of the initial investment of $4,500,000. Monthly or periodic outflow is $100,000 for 6 years or 72 months/periods. Again, in this scenario, because the equipment is paid for upfront, this is the first cash flow included in the calculation. No elapsed time needs to be accounted for, so today's outflow of $4,500,000.00 does not need to be discounted, the two important parameters in this case are the number of periods (t) and the discount rate (r).

- **The Number of periods (t):** The equipment is expected to generate monthly cash flow and last for six years, which means there will be seventy-two cash flows and seventy-two periods included in the calculation. **t=6 x 12=72**
- **Discount rate (i):** The discount rate remains the same as in case I, that is,
  $Monthly\ rate=0.74\%$

<u>Step 2: NPV of future cash flows</u>

<u>Case I:</u> Assume the monthly cash flows are earned at the end of the month, with the first payment arriving exactly one month after the equipment has been purchased. This is a future payment, so it needs to be adjusted for the time value of money. An investor can perform this calculation easily with a spreadsheet or calculator. To illustrate the concept, the first twelve payments are displayed in table 3.

$$\boldsymbol{NPV}=-\boldsymbol{PV}+PVECF,\Rightarrow NPV=\ -\boldsymbol{PV}+\sum_{\boldsymbol{t=1}}^{\boldsymbol{n}}\frac{\boldsymbol{R_t}}{(\mathbf{1}+\boldsymbol{i})^{\boldsymbol{t}}}$$

Where R_t=$85,000 for t=1, 96

$$NPV=-\$4{,}500{,}000+\sum_{t=1}^{96}\frac{85{,}000_{96}}{(1+0.0074)^{96}}$$

$$NPV=-\$4{,}500{,}000+\$5{,}826{,}696.19=\$1{,}326{,}696.19$$

The full calculation of the NPV is equal to the present value of all 96 future cash flows, minus the $4,5000,000.00 investment. The calculation would be more complicated if the equipment is expected have a salvage value greater than zero.

The calculation is easily accomplished using EXCEL. A way to carry out the computation PVECF component of NPV is extremely straightforward as shown in in the following steps:

i. Open an EXCEL page
ii. For column A of the page, make 96 rows and number the Rows 1-96 (Rows A1-A96)
iii. For column B, create the space for 96 empty rows (Rows: B1-B96).

iv. Put your cursor in Column B, Row 1, namely, B1

v. In the function window that appears above, type in the PVECF formula as follows: " =85000/((1.0074)^A1) ", that is : =85000/((1.0074)^A1)) without the quotes, and hit the "return" key.

vi. The cell or Row B1 will have the value: ***$84,375.62***

vii. Now copy the content of Row B1. Put your cursor in Row B2 and scroll all the way to row B96 and hit "enter". Rows B2-B96 will be populated with their appropriate values.

viii. In Row B97, type "=SUM (B1:B96)" that is =SUM (B1:B96) and hit "enter"

ix. The cell or Row B97 will have the value: ***$5,826,696.19*** which is the sum of all the PVECF.

Period	Cash Flow	Present Value of Expected Cash Flow (PVECF)
Period 1	$85,000	$\frac{\$85,000}{(1+0.0074)^1} = \$84,375.62$
Period 2	$85,000	$\frac{\$85,000}{(1+0.0074)^2} = \$83,755.83$
Period 3	$85,000	$\frac{\$85,000}{(1+0.0074)^3} = \$83,140.59$
Period 4	$85,000	$\frac{\$85,000}{(1+0.0074)^4} = \$82,529.87$
Period 5	$85,000	$\frac{\$85,000}{(1+0.0074)^5} = \$81,923.63$
Period 6	$85,000	$\frac{\$85,000}{(1+0.0074)^6} = \$81,321.85$
…….	$85,000	
…….	$85,000	
Period 96	$85,000	$\frac{\$85,000}{(1+0.0074)^{96}} = \$41,882.45$
Total		$\sum_{t=1}^{96} \frac{85,000_{96}}{(1+0.0074)^{96}} = \$5,826,696.19$

Table 3: Computation of PVECF

Case II: Assume the monthly cash flows are earned at the end of the month, with the first payment arriving exactly one month after the equipment has been purchased. This is a future payment, so it needs to be adjusted for the time value of money. An investor can perform this calculation easily with a spreadsheet or calculator. To illustrate the concept, the first eight payments and the last are displayed in the table 4.

$\boldsymbol{NPV} = -\boldsymbol{PV} + PVECF \Rightarrow\Rightarrow NPV = -\boldsymbol{PV} + \sum_{t=1}^{n} \frac{R_t}{(1+i)^t}$, Where R_t=$100,000 for t=1, 72

$$NPV = -\$4,500,000 + \sum_{t=1}^{72} \frac{100,000_{72}}{(1+0.0074)^{72}} = -\$4,500,000 + \$5,561,057.54 = \$1,066,057.54$$

Period	Cash Flow	Present Value of Expected Cash Flow (PVECF)
Period 1	\$100,000	$\frac{\$100{,}000}{(1+0.0074)^1} = \$99{,}265.44$
Period 2	\$100,000	$\frac{\$100{,}000}{(1+0.0074)^2} = \$98{,}536.27$
Period 3	\$100,000	$\frac{\$100{,}000}{(1+0.0074)^3} = \$97{,}812.46$
Period 4	\$100,000	$\frac{\$100{,}000}{(1+0.0074)^4} = \$97{,}093.96$
Period 5	\$100,000	$\frac{\$85{,}000}{(1+0.0074)^5} = \$96{,}380.74$
Period 6	\$100,000	$\frac{\$100{,}000}{(1+0.0074)^6} = \$95{,}672.76$
…….	\$100,000	
…….	\$100,000	
Period 72	\$100,000	$\frac{\$100{,}000}{(1+0.0074)^{72}} = \$58{,}811.17$
Total		$\sum_{t=1}^{72} \frac{100{,}000_{72}}{(1+0.0074)^{72}} = \$5{,}566{,}057.54$
Table 4: Computation of PVECF		

Case III:

The company BIDAN Enterprises Ltd is contemplating setting up a pilot organic poultry farm requiring an initial funds of \$400,000 during the first year. This investment represents an initial cash outlay and is considered a net negative value cash outflow. After the factory is successfully launched in the first year with the initial investment, it would start generation some output in terms of eggs, chicks, etc. by the second year going forward.

That will result in net cash inflows in the form of revenues from the sale of the factory output. As is to be expected, the revenue will increase each succeeding year. The poultry farm generates \$150,000 during year one (second year), \$250,000 in year two, \$320,000 the year three, \$450,000 in year four, and \$580,000 the year five (6th year) when the pilot project is completed, and a new more comprehensive phase is implemented. Table 5 developed in EXCEL shows the initial outflow, the yearly cash inflow, and the corresponding present value (PV)over the pilot period. It is assumed that the WACC is 8% annually. PVECF is present value of expected cash flow

$$\boldsymbol{NPV} = -\boldsymbol{PV} + \mathrm{Sum}(PVECF) = -\$450{,}000 + \$1{,}360{,}535.73 = \$910{,}535.73$$

	A	B	C	D	E	F	G
1							
2		WACC	Year	Year	Cashflows	Formula for PVECF	(PVECF)
3		8%	2022	0	-$450,000	=E3/((1+$B$3)^D3)	-$450,000.00
4		WACC=8%	2023	1	$150,000	=E4/((1+$B$3)^D4)	$138,888.89
5			2024	2	$250,000	=E5/((1+$B$3)^D5)	$214,334.71
6			2025	3	$355,000	=E6/((1+$B$3)^D6)	$281,810.45
7			2026	4	$450,000	=E7/((1+$B$3)^D7)	$330,763.43
8			2027	5	$580,000	=E8/((1+$B$3)^D8)	$394,738.25
						SUM OF PVECF	$1,360,535.73

Table 5: Computation of PVECF

7.4 Return on Investment --ROI

Return on investment, also called the rate of return (RoR)—is the percentage increase or decrease in an investment over a set period. The calculation for the ROI is similar to the RoR except there is no consideration for the time value of money (TVM). ROI can be used in conjunction with the rate of return (RoR), which considers a project's time frame. Since the notion of time value of money (TVM) does not apply, ROI calculated by taking the difference between the current or expected value and the original value divided by the original value and multiplied by 100.

Suppose an investor borrowed $90,000 to start an online business. After 4 years, the business is now valued at $150,000. The ROI is thus: ROI= $\$\left(\frac{150{,}000-90{,}000}{90{,}000}\right)x100 = 66.77\%$

Just like the RoR, the ROI can be negative. Assume that in the earlier example, rather than increase in value, the company is now worth only $50,000 after 5 years, with no prospects for improvement. In this case, the ROI is given as: ROI= $\$\left(\frac{55{,}000-90{,}000}{90{,}000}\right)x100 = -38.89\%,$ which indicates an unfavorable return on investment.

7.5 Rate of Return--RoR

A rate of return (RoR) is the rate of return of a net cash flow and represents net gain or loss of an investment over a specified time period. ROR is time delimited and thus depend on the length of the investment and it is usually stated on an annual basis. The rate of return can be calculated for any investment and any kind of asset. The more pragmatic approach is to measure ROR as that interest rate for which the equivalent benefits equal equivalent costs. Define the following

Equivalent Uniform Annual Benefits = EUAB

Equivalent Uniform Annual Costs =EUAC

The ROR is the interest rate that results in : EUAB – EUAC = 0

Example: Consider the Following cash flow in table 6.

End of Year	Net Cash Flow ($)
0	- 200
1	20
2	20
3	20
4	20
5	220
Table 6: Analysis of RoR	

.

$EUAB - EUAC = 0 = -200(A\,/P, i\%, 5) + 200(A\,/P, i\%, 5) + 20$

The value of the interest rate i that results in EUAB-EUAC=0, is the RoR

For 8%, $(\mathrm{A}\,/\mathrm{P}, 10\%, 5) = 0.2505, (\mathrm{A}\,/\mathrm{P}, 10\%, 5) = 0.1705, \Rightarrow (EUAB - EUAC) = 4)$

For 10%, $(\mathrm{A}\,/\mathrm{P}, 10\%, 5) = 0.2638, (\mathrm{A}\,/\mathrm{P}, 10\%, 5) = 0.1638, \Rightarrow (EUAB - EUAC) = 0)$

Hence RoR for this cash flow is 10%

Another aspect of RoR that is often of interest is the Compound Annual Growth Rate (CAGR). CAGR is the rate of return (RoR) that would be required for an investment to grow from its beginning balance to its ending balance, assuming the profits were reinvested at the end of each period of the investment's life span. It is the measure of an investment's annual growth rate over time, with the effect of compounding considered. It is often used to measure and compare the past performance of investments or to project their expected future returns.

The equation for CAGR is given as $CAGR\% = \left\{\left[\frac{EV}{BV}\right]^{\frac{1}{n}} - 1\right\}x100$ Where:

EV=ending value, BV=beginning value, n =number of compounding periods

End of Year	Ending Value
1	**$2000**
2	**$3500**
3	**$4800**
4	**$5900**
5	**$8000**
Table 7: CAGR Calculation	

$$CAGR\% = \left\{\left[\frac{EV}{BV}\right]^{\frac{1}{n}} - 1\right\}x100 = \left\{\left(\frac{2000}{8000}\right)^{\frac{1}{5}} - 1\right\}x100 = 31.95\% \approx 32\%$$

ROI can be used in conjunction with the rate of return (RoR) as a simple version of RoR. For the CAGR for example, the simple ROI for this problem with respect RoR is:

$[(8000 - 2000)/8000]x100 = 75\%$

7.6 Internal Rate of Return--IRR

The internal rate of return (IRR) is one of many measures used in capital budgeting to estimate the profitability of potential investments. IRR is the interest rate that makes the net present value (NPV), future worth(NFW), and the Annual Worth (NAW) of all cash flows from a particular project or investment equal to zero in a discounted cash flow analysis and utilizes the notion of the time value of money (TVM). To find IRR, the ensuing calculation sets the net present value (NPV), net future worth (NFW) or the net annual Worth (NAW) of the project's future cash flows equal to zero and then solves for the investment's IRR. So, a project's internal rate of return (IRR) is the same whether calculated with the present worth, future worth, or annual equivalent worth methods.

The higher the IRR, the better the investment. The ultimate goal of IRR is to identify the discount rate which makes the present value of the sum of annual (or future) nominal cash inflows equal to the initial net cash outlay for the investment. When this sum is zero, it means that the cash inflows and the cash outlays are equal (i.e., breakeven). IRR is not the actual dollar value of the project but the annual return that makes the sum or the net value equal to zero.

Please note that for the purpose of Engineering Economic Analysis, or economic analysis in general, a project is valid if its internal rate of return (IRR) is greater than or equal to the stated MARR.

This calculation produces a single annual rate of return for an investment, that is: $NPV = 0$, that is: $NPV = 0 \Rightarrow \sum_{t=1}^{T} \frac{C_t}{(1+r)^t} - C_0 = 0$

Where:
- T = total number of time periods
- t = time period counter
- r = Discount Rate
- C_t = Net cash flows during the period t
- C_0 = Total Initial Investment

The value of the discount rate (r)that makes NPV =0 is IRR

Example AW: A 10-year project has an initial cost of $100, 000, net annual cash inflows of $15, 000, and a salvage value of $20,000. The equation for the IRR based on the Annual Worth (AW) is:

$$[-\$100{,}000\ (A/P, \mathrm{i}^*, 10) + \$15{,}000 + \ \$20{,}000(\mathrm{A/F}, \mathrm{i}^*, 10)]\ = 0$$

$$for\ \mathrm{i}^* = 9.97\%, \mathrm{NAW} = \$6.6, \text{for } 9.98\%, \mathrm{NAW} = -\$4\ , for\ \mathrm{i}^* = 9.974\%, \mathrm{NAW} = \$0.4$$

So, IRR≈ 9.974%%

Example for PV: A 10-year project has an initial cost of $100, 000, net annual cash inflows of $15, 000, and a salvage value of $20,000. The equation for the IRR based on the Present Value (NPV) is:

$$[-\$100{,}000\ + \$15{,}000(P/A, \mathrm{i}^*, 10) + \ \$20{,}000(\mathrm{P/F}, \mathrm{i}^*, 10)]\ = 0$$

$$for\ \mathrm{i}^* = 9.97\%, \mathrm{NPV} = \$19.5, \text{for } 9.98\%, \mathrm{NPV} = -\$25.0, for\ \mathrm{i}^* = 9.974\%, \mathrm{NPV} = \$0.2$$

So, IRR≈ 9.974%%

Example for FW: Same problem for future worth

$$[-\$100{,}000\ (F/P, \mathrm{i}^*, 10) + \$15{,}000(\mathrm{F/A}, \mathrm{i}^*, 10) + \ \$20{,}000]\ = 0$$

$$for\ \mathrm{i}^* = 9.97\%, \mathrm{NFW} = \$57.50, \text{for } 9.98\%, \mathrm{NFW} = -\$70,, for\ \mathrm{i}^* = 9.974\%, \mathrm{NFW} = \$2.5$$

So, IRR≈ 9.974%%

7.6.2 NPV and IRR

As part of capital budgeting, NPV, and IRR are among two of the common metrics used to judge the viability of an investment. The IRR method and the present worth method result in the same investment decision whether the projects are independent (standalone) or mutually exclusive (no two can occur at the same time).

7.6.3 Arithmetic Gradient and IRR

We earlier discussed the gradient as a cash flow, a different one, nonetheless. The issue of IRR is also important even with gradients because the idea is that for any investments, the interest rate that renders the cash flow in a NPV (or AW, FW) equal to zero ensures that the investment is at least viable.

Example: A 5-year project has an initial cost of $75,000, net annual cash inflows of $15,000 in the first year subsequently increasing by $2,000 each year. If the project's salvage value is $0.

This is an arithmetic gradient with G=$2,000. A_1=$15,000, P=$75,000. So, the net cash flow or the sum of the initial cash outlay and the cash inflow based on AW is:

For A: $[-\$75{,}000\ (A/P, \mathrm{i}^*, 5) + \$15{,}000 + \ \$2{,}000(\mathrm{A/G}, \mathrm{i}^*, 5) + 0] = 0$

For IRR=7%, NAW=$437.5, for IRR=8%, NAW=-94.50, hence **IRR=7.8%**

If the salvage value is $5,000 after 5 years, then we have

For A: $[-\$75{,}000(A/P, \mathrm{i}^*, 5)\ + \$15{,}000 + \ \$2{,}000(\mathrm{A/G}, \mathrm{i}^*, 5) + \$5000(\mathrm{A/F}, \mathrm{i}^*, 5)] = 0$

We can formulate the problem for P, and F as follows

For P: $[-\$75{,}000\ + \$15{,}000(P/A, \mathrm{i}^*, 5) + \ \$2{,}000(\mathrm{P/G}, \mathrm{i}^*, 5) + 0] = 0$, for salvage value=0

For F: $[-\$75{,}000(F/P, \mathrm{i}^*, 5)\ + \$15{,}000(F/A, \mathrm{i}^*, 5) + \ \$2{,}000(\mathrm{F/G}, \mathrm{i}^*, 5) + 0] = 0$, salvage 0

If the salvage value is $5,000 after 5 years, then we have

For P: $[-\$75{,}000\ + \$15{,}000(P/A, \mathrm{i}^*, 5) + \ \$2{,}000(\mathrm{P/G}, \mathrm{i}^*, 5) + \$5000(\mathrm{P/F}, \mathrm{i}^*, 5)] = 0$

For F: $\left[-\$75{,}000\left(\frac{F}{P}, \mathrm{i}^*, 5\right)\ + \$15{,}000\left(\frac{F}{A}, \mathrm{i}^*, 5\right) + \ \$2{,}000\left(\frac{\mathrm{F}}{\mathrm{G}}, \mathrm{i}^*, 5\right) + \$5000\right] = 0$

7.6.4 Multiple IRR

Multiple IRR occur typically when we have nonconventional cash flows. That is, when the direction of the cash flows switches more than once there is a very strong likelihood of multiple IRR. This is based on the rule of signs proposed by the16th century French philosopher, scientist, and mathematician René Descartes. Descartes' rule of sign asserts that the number of positive real zeros in a polynomial function corresponds approximately to changes in the sign of the coefficients. Stated differently, the number of positive roots is at most the number of sign changes in the sequence of polynomial's coefficients (omitting the zero coefficients. For example, where we have cash flows with negative cash flow, then positive cash flow, and then negative cash flow values such as (-+-), then hypothetically we can have up to two or more IRR. Such a cash flow profile is called non-conventional because the direction switches more than one time. Thus, a cash flow with two or more sign changes will likely have more than one IRR.

However, most investments will have a cash flow structure which excludes multiple IRRs. If a project is a simple investment, it will have at most one positive IRR.

When we plot the discount rate versus the NPV profile, then the IRR is the intersection of the NPV profile and the X-axis. In other words, it is the zero of the NPV. Hence, the IRR is the value of the discount rate at which NPV is zero

A conventional or simple investment is an investment characterized by one or more periods of cash outflows, followed by one or more periods of cash inflows as shown in figure 13. In figure 13, and table 8, there is only one direction change in the cash flow, namely, from time 0 to time 1 and in such a case, we will expect a single IRR.

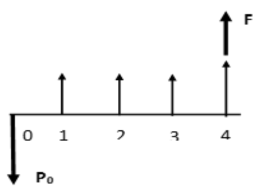

Figure 13: Cash Flow Diagram for a Simple Investment

Time	Cash Flow
0	- $40,000
1	+$10,000
2	+$15,000
3	+$10,000
4	+$20,000

Table 8. Cash Flows for a Typical Conventional Investment

Based on the Figure 12 and Table 6, we compute the NPV as follows:

$$NPV = -\$40K + \$10K(1+i)^{-1} + \$15K(1+i)^{-2} + \$10K(1+i)^{-3} + \$20K(1+i)^{-4}$$

For i=2%, NPV=$12,122. 585
For i=10, NPV=$2,661.020
For i=12.7725%, NPV=$0

Also, using EXCEL, we compute a series of values of NPV corresponding to different discount rates (Figure 14). The plot of the discount rate versus NPV (Figure 13) confirms that we only have one IRR which is 12.7% in this case

For nonconventional investment, the cash flow switches direction two or more times and based on Descartes' rule of sign, we should expect the number of IRR to be at least 2. In figure 15, we have a nonconventional cash flow, where the cash flow switches directions 2 times. This means that we can expect the IRR=2. This means that we have two possible solutions for IRR.

Discount Rate	NPV
0.00%	$15,000
2.00%	$12,122
2.50%	$11,438
3.00%	$10,769
3.50%	$10,113
3.59%	$10,000
4.00%	$9,470
6.00%	$7,022
8.00%	$4,758
10.00%	$2,661
12.00%	$715
12.77%	$0
16.00%	($2,779)
18.00%	($4,351)
20.00%	($5,818)
22.00%	($7,190)

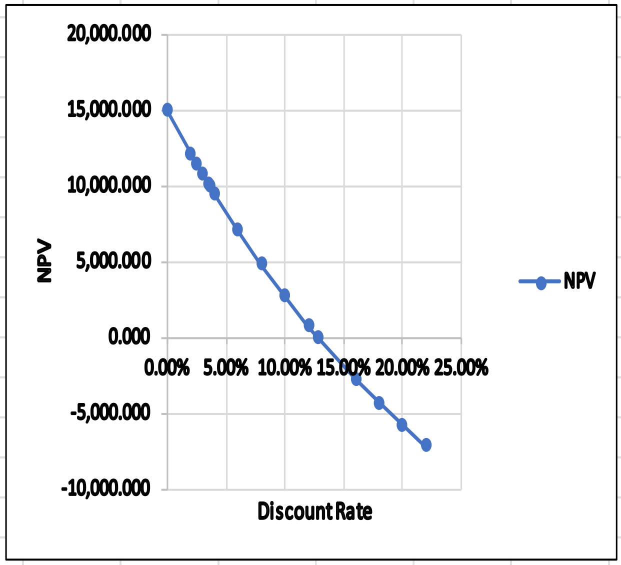

Figure 14: Plot of NPV versus Discount Rate

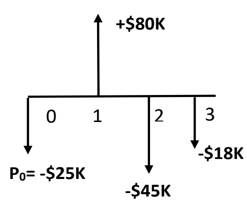

Figure 15: Cash Flow Diagram for a Simple but Non-Conventional Investment

Time	Cash Flow
0	- $25,000
1	+$80,000
2	-$45,000
3	-$18,000

Table 9. Cash Flows for a Typical Conventional Investment

Based on the Figure 15, and Table 9, we compute the NPV as follows:

$NPV = -\$25K + \$80K(1+i)^{-1} - \$45K(1+i)^{-2} - \$18K(1+i)^{-3}$

For i=3%, NPV = - $6,497.66
For i=10%, NPV = - $1,008.65
For i=20%, NPV = $0
For i=115%, NPV = +$663.15
For i=120%, NPV = +$375.66, For i=126%, NPV ≈ $0

Also, using EXCEL, we compute a series of values of NPV corresponding to different discount rates (Figure 16) . The plot of the discount rate versus NPV (Figure 16) shows that the NPV profile crosses the X-axis at two points. Hence the zeros of the NPV function are:
$\langle i = 20\%, 126\%\rangle$

Discount Rate	NPV
0%	($8,000)
3%	($6,498)
10%	($2,986)
14%	($1,600)
16%	($1,009)
20%	$0
25%	$984
30%	$1,718
45%	$2,865
50%	$3,000
60%	$3,027
70%	$2,824
98%	$1,607
100%	$1,500
110%	$948
115%	$663
120%	$376
126%	$0
150%	($1,352)
170%	($2,458)

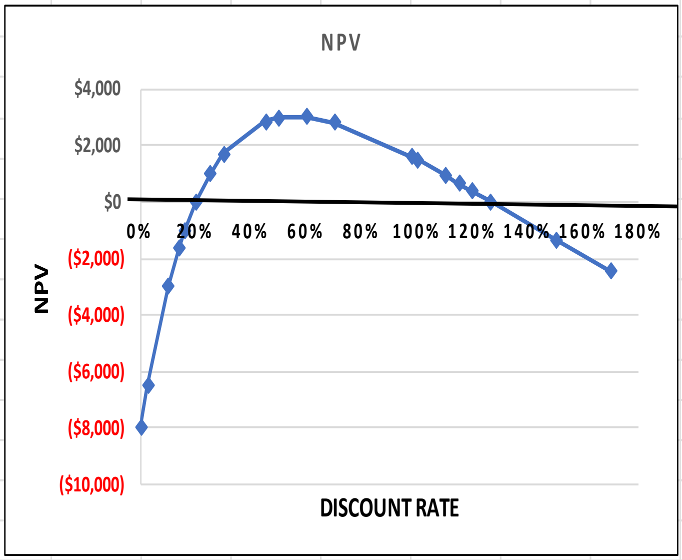

Figure 16: Plot of NPV versus Discount Rate

7.7 External Rate of Return--ERR

The **External Rate of Return** (ERR) is the ROR on a project where any excess cash from a project is assumed to earn interest at a pre-determined explicit rate which is typically the Minimum Acceptable ROR (MARR). With ERR, the revenue generated, or the profit earned cannot be plowed back into the same project but rather it is reinvested elsewhere. It is the interest rate external to a project at which the net cash flows generated or required by the project over its life can be reinvested or borrowed, not for the same project but a different project.

The idea of ERR is anchored around an external reinvestment interest rate i_{ERR}, where i_{ERR} is the interest that the cash flows generated by the project is reinvested in another project. It is what you do with the cash that is generated from the project that determines what the ERR is about.

For example, assume that an imaginary investor who does not like or play baseball bought 10 of Mickey Mantle's Autographed Official AL Baseball New York Yankees single signed homerun baseball for $200 each in 1961. In 2020, each homerun baseball expectedly sold for $350,000. The investor decides to cash out his investment and invest it in one of the leading electric car battery manufacturer, LG Energy solutions. In this case our investor is buying something else with the profit and not reinvesting it in baseball paraphernalia. This is an example of ERR, when revenue cannot be reinvested back into the project but invested elsewhere.

We know from our previous example in Figure 16 that multiple IRRs are possible. Unfortunately, it is sometimes difficult to know in advance when there are multiple IRRs. However, most investments will have a cash flow structure which excludes multiple IRRs such as the one we described as simple and conventional investment which had only one change in sign. If a project is a simple investment, it would have at most one positive IRR. As explained earlier, we know from Descartes rule of signs that the number of positive real zeros in a polynomial function corresponds approximately to the changes in the sign of the coefficients. This means that when there are cash flow with repeated changes (up and down), then there is a strong likelihood of multiple IRRs. As a general rule, the ERR method is recommended whenever multiple IRRs are possible.

7.7.1 Difference between ERR and IRR

Rate of return(ROR) is a profit on an investment over a period of time, expressed as a proportion of the original investment. **Internal Rate of Return**(IRR) is obtained by calculating the ROR with only internal factors without considering external factors such as inflation or cost of capital. Generally speaking, the higher a project's internal rate of return, the more desirable it is to undertake. Assuming the costs of investment are equal among the various projects, the project with the highest IRR would probably be considered the best and be undertaken first. IRR is the rate of growth a project is expected to generate. While the actual rate of return that a given project ends up generating will often differ from its estimated IRR, a project with a substantially higher IRR value than other available options would still provide a much better chance of strong growth.

IRR is most often to evaluate or rather compare the profitability of proposed new projects versus the expansion or rejigging of existing ones. For example, an energy company may use IRR in deciding whether to open a new power plant or to renovate and expand a previously existing one. While both projects are likely to add value to the company, it is likely that one will be the more logical decision as prescribed by IRR.

The ERR is the ROR on a project where any excess cash from a project is assumed to earn interest at a pre-determined explicit rate —usually the Minimum Acceptable ROR (MARR). Computing an exact ERR is difficult hence we talk of approximate ERR. If all net project receipts are taken forward at the MARR to the time of the last cash flow and all net project disbursements are taken forward at an unknown interest rate to provide the future worth (equal 0), the unknown interest rate that enables that equality that take place is the approximate ERR. The formula for computing ERR is given as:

$$\sum_{k=0}^{n} E_k(P/F, MARR\%, k)(F/P, ERR\%, n) = \sum_{k=0}^{n} R_k(P/F, MARR\%, n-k), \text{ OR}$$

$$\sum_{k=0}^{n} E_k(P/F, MARR\%, k) = \sum_{k=0}^{n} R_k(P/F, MARR\%, n-k)(P/F, ERR\%, n), \text{ where}$$

$R_k = Excess\ receipts\ \ over\ and\ above\ the\ expenses\ in\ period\ k$

$E_k = Excess\ expenditures\ over\ and\ above\ receipts\ \ in\ period\ k$

n= number of periods, MARR=Maximum Attractive Rate of Return, ERR=External Rate of Return

The essence of the formula is demonstrated in figure 17. The idea of the excess receipts or excess expenditures is especially important when the cash inflows (plus sign) and cash outflows (negative sign)are coincident, namely they occur at the same time period. When that is the case, the cash flows are added. If the result from the addition is positive then the resultant cash flow at that time period is a cash inflow and it will have a positive sign. If the result of the addition is negative, then the resultant cash flow at that particular period is a cash outflow and will have a negative sign.

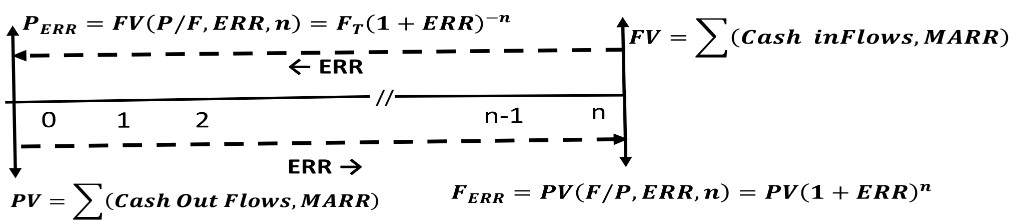

Figure 17: Estimating ERR using Excess Cash Inflows and Excess Cash Outflows

Example: Suppose we have the cash flow for an investment as shown on table 10 with the cash flow diagram shown on figure 18. Assume MARR=10%

Time	Cash Flow
0	-8
1	6
2	2
3	-7
4	8

Table 10: Cash Flows for Non-Simple or Non-conventional Investment

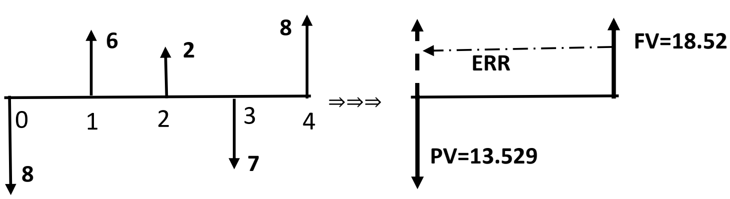

Figure 18: Cash Flow Diagram for Non-conventional Investment

Inflow: $FV = 6(F/P, 10\%, 3) + 2(F/P, 10\%, 2) + 8 = 6(1.35) + 2(1.221) + 8 = 18.52$

Outflow: $PV = 8 + 7(P/F, 10\%, 3) = 8 + 7(0.7513) = 8 + 5.259 = 13.529$

$\mathrm{P_{ERR}} = \mathrm{FV(P/F, ERR, 4)} \Rightarrow \mathrm{PV} = \mathrm{FV}(1+\mathrm{ERR})^{-4}$

$$\mathrm{PV} = \mathrm{FV}(1+\mathrm{ERR})^{-4} \Rightarrow 13.259 = 18.52(1+\mathrm{ERR})^{-4} \Rightarrow (1+\mathrm{ERR})^4 = \frac{18.52}{13.259} = 1.3968$$

$$(1+\mathrm{ERR})^4 = 1.3968 \Rightarrow \left[(1+ERR) = (1.3968)^{\frac{1}{4}}\right] \Rightarrow (1+ERR) = 1.0871$$

Hence: $ERR = (1.0871 - 1) = 8\% < MARR\ (10\%). Investment\ not\ Viable$

Example: Determine the ERR for the following cash-flow, for an external reinvestment rate of 3.5%
It is suggested that you use MS Excel for this problem.

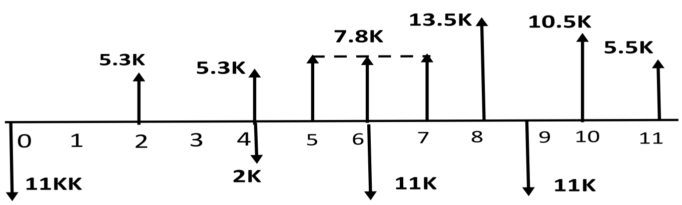

Figure 19: Cash Flow Diagram for Non-conventional Investment

MARR=0.035						
YEAR	**YEARS UNTIL YEAR 11**	**AMOUNT**	**NET OUTFLOW**	**NET INFLOW**	**PV of outflow at (i=MARR)**	**FV of inflow at (i=MARR)**
0	**11**	**-11000**	**11000**	**0**	**$11,000.00**	**$0.00**
1	**10**	**0**	**0**	**0**	**$0.00**	**$0.00**
2	**9**	**5300**	**0**	**5300**	**$0.00**	**$7,223.36**
3	**8**	**0**	**0**	**0**	**$0.00**	**$0.00**
4	**7**	**3300**	**0**	**3300**	**$0.00**	**$4,198.52**
5	**6**	**7800**	**0**	**7800**	**$0.00**	**$9,588.19**
6	**5**	**-3200**	**3200**	**0**	**$2,603.20**	**$0.00**
7	**4**	**7800**	**0**	**7800**	**$0.00**	**$8,950.68**
8	**3**	**13500**	**0**	**13500**	**$0.00**	**$14,967.69**
9	**2**	**-11000**	**11000**	**0**	**$8,071.04**	**$0.00**
10	**1**	**10500**	**0**	**10500**	**$0.00**	**$10,867.50**
11	**0**	**5500**	**0**	**5500**	**$0.00**	**$5,500.00**
			SUM=		**$21,674.24**	**$61,295.94**
ERR is the dscount rate at which the FV of the inflow is equal to the PV of the Outflow						
		Sum of PV=$21,674.24				**Sum of FV=$61,295.94**
					ERR =	**0.10**
						$21,674.24

Table 11: Computation of ERR

Note. When the when the cash flows are coincident, they are added. If the value is positive then the cash flow for the period is considered an inflow. If it is negative, than it is considered and outflow. Please confirm that this is the case using the cash flow diagram(figure 19) and table 11.

From table 11, FV for inflows =$61,295.94

From Table 11, PV for cash inflows=$21,674.24

$\sum_{k=0}^{n} E_k(P/F, MARR\%, k) = PV = \$61,295.94,$

$\sum_{k=0}^{n} R_k(P/F, MARR\%, n-k) = FV = \$21,674.24$

$$\sum_{k=0}^{n} E_k(P/F, MARR\%, k)\,(F/P, ERR\%, n) = PV(F/P, ERR\%, n)$$
$$PV(F/P, ERR\%, n) = PV(1+ERR)^n = \$61{,}295.94(1+ERR)^{11}$$
$$Recall: F = P(1+i)^n \Rightarrow [FV = PV(1+ERR)^n] \Rightarrow \; [PV(1+ERR)^{11} = FV]$$
$$\frac{FV}{PV} = (1+ERR)^{11} = \frac{61{,}295.94}{21{,}674.24}$$
$$(1+ERR) = \left(\frac{61{,}295.94}{21{,}674.24}\right)^{\frac{1}{11}} \Rightarrow (1+ERR) = 1.0991$$
$$(ERR) = 1.0991 - 1 = 0.0991 \; \approx 10\%$$

Since ERR (=10%)> MARR(=3.5%) then the investment is viable and so should be funded.

Note the value of 10% on the EXCEL table was obtained by using the EXCEL function "GOAL SEEK" which a search optimization function under "What if Analysis" utility which is located under the "Forecast" window, under the Data Panel of EXCEL

To use GOAL, SEEK, you need to do the following as shown in table 11.

1. Compute the FV of each of the inflows using MARR and obtain the total Sum
2. Compute the PV of each of the cash outflows using MARR and obtain the total Sum
3. Make a reasonable guess of the value of ERR. For this problem, the initial guess was 0.08
4. Compute the PV of the sum of the FV computed in 1 above using the value of ERR in 3. Note that the PV obtained in 4 will be close to the PV in 2 if your guess is very good.
5. The goal is to make the PV computed in 4 equal to the PV computed in 2. So, the value of ERR that makes the two quantities equal is the desired ERR.
6. The purpose of GOAL SEEK is to help search for the value of ERR that would make the PV of the sum of the FV of the inflow equal to the PV of the outflow
7. To get GOAL SEEK you first click on Data panel on the main EXCEL panel. It will open up several windows, one of which is forecast. Under forecast is "What If". Pull down the "What if" window and you will find three objects the middle of which is GOAL SEEK. GOAL SEEK has three entries that you need to fill in.
    a. The first is "Set Cell". This represents the cell containing the value of the PV of the FV of the inflow that you just computed in 4. Click on the cell containing that value to put it into GOAL SEK. Do NOT manually type it
    b. The second is "To Value". Manually type in the PV of the outflow computed in 2.
    c. The third is "Change Cell". This is the cell containing the guessed at value of ERR that you want to change to the approximate value of ERR. Click on that cell so that GOAL SEEK to acquire the value.
    d. After filling in the entries, hit enter. If the search is successful GOAL seek will replace the ERR you guessed with the ERR that would make the PV of the FV of the inflow

equal to the PV of the outflow. It would also replace the value of the PV of the FV of the inflow with a value that is equal to the PV of the outflow as shown on the table 11.

Please note that the approximate ERR will always be between the precise ERR and the MARR. The underlying assumption about the precise ERR is that all project receipts are reinvested at the MARR when the project balance is positive only.

With respect to the relationship between IRR and ERR, please note as follows

1. You may use IRR for simple investments or investments for which you have ruled out the existence of multiple IRR
2. If you know or if you have found multiple IRR by plotting NPV (or PW) versus different discount rates, then you can compute approximate ERR for the investment.
3. In terms of investment decisions, approximate ERR will produce results that are consistent with exact ERR and NPV(or PW).
4. If ERR=IRR, then ERR will produce a result similar to MARR. However, if ERR ≥IRR then the investment can be considered as economically viable

7.8 Profitability Index

Profitability Index (PI) is the ratio of payoff to investment of a proposed project and represents the relationship between the costs and benefits of a proposed project. Also known as profit investment ratio or value investment ratio, it is an appraisal technique used in capital budgeting to evaluate the economic viability of potential capital outlays. It is the ratio between the present value of future expected cash flows and the initial amount invested in the project. A higher PI means that a project will be considered more attractive.

PI measures the monetary benefits (from cash inflows) received for each dollar invested (from cash outflow), with the resultant cash flows discounted back to the present. It compares the present value (PV) of future cash flows received from a project to the initial cash outflow (investment) required to fund the investment

PI is a useful tool for ranking potential investments or projects because it quantifies the value of such investment. Ideally, a PI index should be greater than or equal to unity. An index of 1.0 is the lowest acceptable index because any lower value is an indication that the present value (PV) of the investment is less than the initial investment. Values of PI higher than 1.0 is an indication that the future anticipated discounted cash inflows of the project are greater than the anticipated discounted cash outflows. So, as the value of PI increases, so does the economic viability or financial acceptability of the proposed investment. PI is computed as follows.

$$\text{PI} = (\text{PV of future cash flows}) \div (\text{Initial investment})$$

<u>Example</u>: Table 12 shows, the cash flows of a certain investment through 8 years at a discount rate (MARR) of 12%. The initial investment is \$60K, and the annual Cash inflow is \$15k per year. Find PI?

	MARR	0.12		
Year	**Cash flow**	**1/((1+1)^n)**	**PV of Cash Flow**	**Cumulative PV of Cash Flows**
0	-$60,000.00	1.000	-$60,000.00	-$60,000.00
1	$15,000.00	0.893	$13,392.86	$13,392.86
2	$15,000.00	0.797	$11,957.91	$25,350.77
3	$15,000.00	0.712	$10,676.70	$36,027.47
4	$15,000.00	0.636	$9,532.77	$45,560.24
5	$15,000.00	0.567	$8,511.40	$54,071.64
6	$15,000.00	0.507	$7,599.47	$61,671.11
7	$15,000.00	0.452	$6,785.24	$68,456.35
8	$15,000.00	0.404	$6,058.25	$74,514.60

Table 12: Computation for Profit Index -PI

$PI = (\$74{,}514.60 \div \$60{,}000) = 1.24$

With a PI greater than 1.0 (PI=2.24), this project is deemed as a good investment. Of course, higher values of PI mean more attractive investments.

7.9 Payback Period

The Payback Period is important in Capital budgeting because it measures the amount of time required to recoup the cost of an initial investment via the cash flows generated by the investment. It shows how long it takes for a business to recoup an investment. This type of analysis allows firms to compare alternative investments because companies want to recoup their investment sooner rather than later. Simply put, it is the length of time an investment reaches a breakeven point. Companies and individuals most certainly invest their money to get paid back, which is why the payback period is important when considering investment portfolios. In essence, the shorter payback an investment has, the more attractive that investment becomes to the investor.

The simple payback period or non-discounted payback period disregards the time value of money and is determined by counting the number of years it takes to recover the funds invested. For example, consider. The formula for the Payback is given by:

$$Payback\ Period = \frac{\text{Initial Investment}}{Annual\ or\ Periodic\ Cash\ Flow}, \text{ for equal cash inflow}$$

$$Payback\ Period = \left[Years\ Before\ Breakeven + \frac{Unrecovered\ Amount}{Cash\ Flow\ in\ Recovering\ Year}\right], for\ unequal\ cash\ flow$$

As an example, suppose an investor invests $200,000 today in a project (Project 1) with the expectation that it would generate cash inflow of $40,000 for four years followed by $30,000 per year for two additional years, and $15,000 in years seven through eight. The layout of the cash flows/year based on the project's life is:

($40,000 + $40,000+ $40,000+ $40,000 +$30,000+$30,000, $15,000, $15,000).

If we lay out the cash flows so that the cash flows equal to the initial investment we have:

($40,000 + $40,000+ $40,000+ $40,000 +$30,000+$10,000). This means that the breakeven will occur sometime between year 5 and year 6. We note that in year 6, $30,000 per year is the same as $2,500 per month and so it takes 4 months to accumulate $10,000. Hence, the payback period is 5 years and 4 months. If we use the formula for unequal cash flows, we will have

Year before breakeven= year 5 (Cumulative Cash flow up to Year 5 =$190,000)

Unrecovered amount= $10,000

$$Payback\ Period = \left[5 + \frac{\$10{,}000}{\$30{,}000}\right] = \left(5 + \frac{1}{3}\right) Years = 5\ years\ and\ 4\ months$$

The total cash inflow for the eight years will be ($40K*4+$30K*2+$15K*2)=$250,000

Consider another project (Project 2) that costs $500,000 but will result in cash flows of $250,000 each year for the next 10 years resulting in at $2.5 million total inflow. Using the formula for equal cash flows we have: $Payback\ Period = \frac{\$500{,}000}{\$250{,}000} = 2\ years$

Clearly, the Project 2 would cost the company ten times more money, but it would take longer to recover the investment. Based on the simple payback period, Project 2 would pay back the investment in less time, namely in 2 years as opposed to 5 years and 4 months in the case of the Project 1, which makes the company's earning potential much greater. Based solely on the payback period method, the Project 2 is a better investment.

The drawback of non-discounted Payback period, unlike other methods of capital budgeting, is that it ignores the time value of money (TVM), namely the idea that money is worth more today than the same amount in the future because of the earning potential of the present money.

Most capital budgeting tools such as the Net Present Value (NPV), and IRR, consider the TVM. However, because of its simplicity, both simple Payback is often used especially in making snap decisions about potential investments. However, the fact that the non-discount Payback approach ignores the time value of money renders its results be too optimistic while the discounted Payback approach is more practical and realistic. As a result, corporations tend to use the discounted Payback approach.

7.9.1 Determining the Discounted Payback Period

To determine the discounted payback period, first the periodic cash flows of an investment must be computed. These cash flows (actually future cash flows) are then discounted using the present value factor to reflect the discounting process. The starting with the initial investment, the future discounted cash inflows are progressively added to the initial investment outflow through the investment period. Then discounted payback period process is applied to each additional period's cash inflow to determine the breakeven point, namely, the point at which the cumulative inflows equal the initial investment or outflow. At this point, the project's initial cost has been paid off, with the payback period being reduced to zero. Table 13 gives the initial investment of -$60,000 in year 0 and annual inflow of $15,000 for years 1 through 8. We have MARR of 12% and 15%

<u>Note</u>: If MARR > 0 percent, then the time required to recover a project's initial cost would be shorter with the simple payback method than with the discounted payback method because of TVM and the effect of compounding used in the discounted payback method.

	MARR	0.12			MARR	0.15	
Year	Cash flow	1(/(1+i)^n)	PV of Cash Flow	Cumulative PV of Cash Flows	1/(1+i)^n	PV of Cash Flow	Cumulative PV of Cash Flows
0	-$60,000.00	1.000	-$60,000.00	-60,000.00	1	-$60,000.00	-$60,000.00
1	$15,000.00	0.893	$13,392.86	13,392.86	0.870	$13,043.48	$13,043.48
2	$15,000.00	0.797	$11,957.91	25,350.77	0.756	$11,342.16	$24,385.63
3	$15,000.00	0.712	$10,676.70	36,027.47	0.658	$9,862.74	$34,248.38
4	$15,000.00	0.636	$9,532.77	45,560.24	0.572	$8,576.30	$42,824.68
5	$15,000.00	0.567	$8,511.40	54,071.64	0.497	$7,457.65	$50,282.33
6	$15,000.00	0.507	$7,599.47	61,671.11	0.432	$6,484.91	$56,767.24
7	$15,000.00	0.452	$6,785.24	68,456.35	0.376	$5,639.06	$62,406.30
8	$15,000.00	0.404	$6,058.25	74,514.60	0.327	$4,903.53	$67,309.82

Table 13: Computation of Payback Period

$$Payback\ Period = \left[Years\ Before\ Breakeven + \frac{Unrecovered\ Amount}{Cash\ Flow\ in\ Recovering\ Year}\right]$$

$Payback\ Period = 5 + [(60{,}000 - 54{,}071.64) \div 7{,}599.47] = 5.78\ years$, with MARR =12%

$Payback\ Period = 6 + [(60{,}000 - 56{,}767.24) \div 5{,}639.06] = 6.573\ years$, with MARR =15%

For the simple (undiscounted) Payback Period, $Payback\ Period = \frac{\$60{,}000}{\$14{,}000} = 4\ years$

For this example, the MARR(or discount rate) is 12% with a discounted payback period is of 5.8 years. With discount rate of 15%, the discounted payback period is 6.6 years if the discount rate is 15%. But the simple payback period is 4 years in both cases. This means that as the discount rate, MARR increases, the difference in payback periods of the discounted pay period and simple payback period increases.

So, in general, for MARR>0, then the time required to recover a project's initial cost will be shorter with the simple payback method than with the discounted payback method

The payback period approach is of interest in equipment replace strategies because of the significance of the payback period for equipment with specific cash flow profiles as part of the capital budgeting decision. Whether for alternatives or independent equipment, the payback period analysis is an important component of any equipment replacement decisions.

Example:

Due to increasing demand, a manufacturing company is considering replacing its existing machine with new fully automated CNC equipment. Three popular models (DMG-machine A, FANUC -machine B, and MAZAK-machine C) are available at the cost of $ 60k, $62k, and $65k for machines A, B, and C respectively. The salvage value of the existing machine is estimated at $90k. The auxiliary electrical/electronic infrastructure of the existing equipment can be upgraded to support Machine A at an additional cost of $110k. For machine B, the existing electrical, cabling, and other support structure can be used but would require an additional expense of $120k for renovations. In the case of machine C, a new support platform would be required at a cost of $140k due to its unique, cabling, electrical and electronic requirements. If the cabling, electrical and other support platform for the old machine is entirely scrapped, the salvage value is $40,000. The expected cash flows for the three machines are as shown in table 14 and 15. Assume a discount rate MARR of 12%. Determine the Payback period for each machine type. The expected salvage for the new machines is $60k, $62k, $65k respectively.

Particulars	A	B	C
Cost of machine	$ 600,000	$ 620,000	$ 650,000
Cost of Support renovations	$ 110,000	$ 120,000	$ 140,000
Salvage of old machine	($ 90,000)	($ 90,000)	($90,000)
Salvage of old machine			($ 30,000)
Total Exp	$ 620,000	$ 650,000	$ 670,000

Table 14: Project Data Summary

Year	P/F factor @12%	Cash inflows (Mach A)	PV (Mach A)	Cum PV (Mach A)	Cash inflows (Mach B)	PV (Mach B)	Cum. PV (Mach B)	Cash inflows (Mach C)	PV (Mach C)	Cum. PV (Mach C)
0	1.000	$620,000	$620,000	$620,000	$650,000	$650,000	$650,000	$670,000	$670,000	$670,000
1	0.893	$120,000	$107,143	$107,143	$220,000	$196,429	$196,429	$300,000	$267,857	$267,857
2	0.797	$170,000	$135,523	$242,666	$240,000	$191,327	$387,755	$320,000	$255,102	$522,959
3	0.712	$195,000	$138,797	$381,463	$210,000	$149,474	$537,229	$330,000	$234,887	$757,847
4	0.636	$220,000	$139,814	$521,277	$200,000	$127,104	$664,333	$290,000	$184,300	$942,147
5	0.567	$260,000	$147,531	$668,808	$162,000	$91,923	$756,256	$265,000	$150,368	$1,092,515

Table 15: Discounted Payback Period Computation for Machine Replacement/Selection

$$Payback\ Period = \left[Years\ Before\ Breakeven + \frac{Unrecovered\ Amount}{Cash\ Flow\ in\ Recovering\ Year}\right]$$

<u>**For machine A**</u>:

$$Payback\ Period = \left[4 + \frac{\$(620{,}000 - 521{,}277)}{\$147{,}531}\right] = (4 + 0.67) = 4.67\ years$$

<u>**For machine B**</u>:

$$Payback\ Period = \left[3 + \frac{\$(650{,}000 - 537{,}229)}{\$127{,}104}\right] = (3 + 0.89) = 3.89\ years$$

<u>**For machine C**</u>:

$$Payback\ Period = \left[2 + \frac{\$(670{,}000 - 522{,}959)}{\$255{,}102}\right] = (2 + 0.58) = 2.58\ years$$

The results show that although Machine C is the most expensive, it has the lowest payback period of about 2 and half years. Hence, if no other analysis were conducted, and if these alternatives are independent (for example buying one does not preclude buying any others) and if the decision has to be based strictly on the payback period, then Machine C would be selected. Please note that in the case where the projects are mutually exclusive alternatives which requires only one of the alternative to be selected, then further analyses are usually performed to ensure that indeed the choice is correct. One such analysis is the incremental analysis which we will get into later in the follow up course.

7.10 Benefit-Cost Analysis

Benefit-Cost Analysis (BCA) is a method that determines the economic viability of a project by comparing the cash flow stream of benefits and costs over the duration of a project or investment. reduction benefits of a hazard mitigation project and compares those benefits to its costs. It is a systematic tool used to compare the benefits and costs of a project to determine if the proposed investment is sound.. A project is considered cost-effective when the Benefit-Cost ratio (BCR) is ≥ 1.0. Define B as benefit, and C as Cost, then: $BCR = [(PW\ Benefits) \div (PW\ Costs)]$

$$IF: \begin{Bmatrix} BCR = 1 & Breakeven\ Investment \\ BCR = \geq 1 & Profitable\ Investment\ (B > C) \\ BCR = < 1 & Investment\ not\ attractive\ (B < C) \end{Bmatrix}$$

Benefit-Cost Analysis is typically used to compare mutually exclusive alternative investments. In that type of scenario, the analysis is a lot more detailed and is based on incremental analysis. We will explore BCR for mutually exclusive alternative in more details in the not course sequence. Suffice it to say that BCR is a powerful for analyzing mutually exclusive investments.

<u>Example</u> . The cash flow shown in figure 20 is about a project consisting of a simple one project investment. Determine if this is an acceptable project based on the BCR and MARR of 8%

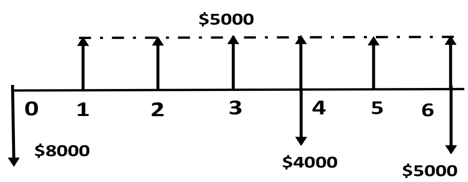

Figure 20: Cash Flow Diagram for Non-conventional Investment

$PW_B = A[P/A, 8\%, 6] = A\left[\frac{(1+i)^n - 1}{i\,(1+i)^n}\right] = \$5000\left[\frac{(1.08)^6 - 1}{0.08\,(1.08)^6}\right] = \$5000[4.6229] = \$23{,}114.40$
$P = F(1+i)^{-n} \Rightarrow (P/F) = (1+i)^{-n}$
$PW_C = \$8000 + F[P/F, 8\%, 4] + F[P/F, 8\%, 6]$
$PW_C = \$8K + \$4K[(1.08)^{-4}] + \$5K[(1.08)^{-6}] = \$8K + \$4K(0.735) + \$5000(0.630)$
$PW_C = \$8000 + \$2{,}940.119 + \$3{,}250.848 = \$14{,}190.967$
$BCR = (PW\ Benefits \div PW\ Costs)(= \$23{,}114.400 \div \$14{,}190.967) = 1.62$
<u>Since BCR >1</u>, we will conclude that the project is viable and hence should be executed.

<u>Example:</u> Pasco County Florida was recently awarded some funds from the recently passed Federal Infrastructure Bill and is considering some road improvements between Land O' Lakes and New Port Richey along the State Route 54 corridor. Four local contractors have submitted bids, and each contractor is guaranteeing (based on county requirements) that the improvements would last for at least

35 years. Some of the reason for the road improvement include the following: a) accident reduction, b) reduced travel time, c) costs to operate the highway including lighting and related electronics, and d) routine maintenance. The discount rate used by this county for this type of project is 5%. Table 16 is the bid data from the county procurement office.

			Proposed Annual Benefits		
Contractor #	Construction Cost	Annual Routine Maintenance	Travel Time Reduction	Operating Costs Reduction	Savings Due to Reduced Accidents
A	$1,650,000	$42,000	$60,000	$30,000	$80,000
B	$1,700,000	$46,000	$70,000	$30,000	$80,000
C	$1,900,000	$40,000	$81,000	$36,000	$80,000
D	$1,855,000	$51,000	$80,000	$35,000	$85,000

Table 16: Benefit-Cost Analysis for Road Improvement

Discount rate =5%, n=35 years

$(P/A, 5\%, 35\,)=\left[\frac{(1.05)^{35}-1}{(0.05)(1.05)^{35}}\right] = 16.3742$

A: Benefit: $= (\$60K + \$30K + \$80K)(P/A, 5\%, 35\,) = \$170k[16.3742] = \$2{,}783{,}614$

Cost: $= \$1.65m + \$42k[16.3742] = \$2{,}337{,}716.40$

B: Benefit: $= (\$70K + \$30K + \$80K)(P/A, 5\%, 35\,) = \$180k[16.3742] = \$2{,}947{,}356$

Cost: $= \$1.7m + \$46k[16.3742] = \$2{,}453{,}.20$

C: Benefit: $= (\$197K)(P/A, 5\%, 35\,) = \$197k[16.3742] = \$3{,}225{,}717.40$

Cost: $= \$1.9m + \$40k[16.3742] = \$2{,}554{,}968.00$

D: Benefit: $= (\$200K)(P/A, 5\%, 35\,) = \$200k[16.3742] = \$3{,}274{,}840.00$

Cost: $= \$1.855m + \$51k[16.3742] = \$2{,}690{,}084.00$

		(P/A, 0.05,35)=	16.3742		
Contractor	Benefit/ COST	PV	AW	PW	PW(B)÷PW(C)=BCR
A	B		$170,000.00	$2,783,614.00	(2783614÷2337716)
	C	$1,650,000.00	$42,000.00	$2,337,716.40	1.191
B	B		$180,000.00	$2,947,356.00	(2947356÷2453213)
	C	$1,700,000.00	$46,000.00	$2,453,213.20	1.201
C	B		$197,000.00	$3,225,717.40	(3225717÷2554968)
	C	$1,900,000.00	$40,000.00	$2,554,968.00	1.263
D	B		$200,000.00	$3,274,840.00	(3274840÷2690084.2)
	C	$1,855,000.00	$51,000.00	$2,690,084.20	1.217

Table 17: Benefit Cost Ratio (BCR) Computation for the Road Improvement Data

From table 17, the RCR value for all the contractor is greater than 1. This simply says that all the projects are viable. Please note that the purpose of this exercise is to demonstrate how we go about computing BCR and not necessarily to select the most viable project. Although Contractor C has the highest BCR ratio, it may or may not be the best candidate since these are mutually exclusive but not independent alternatives. Please note that the county could, based on the BCR ratio alone, select contract C which has the highest BCR value. This is a good solution and could turn out to be the optimal solution, but it is difficult to say without further analysis. With respect to BCR, all the contractors meet the minimum requirements because the BCR values are all ≥1.0

To determine the best candidate would require additional computation based on incremental analyses in which the alternatives are ranked in ascending order of cost and pairs of alternatives are then compared based on their differences in benefit and cost, with higher cost alternative serving as the challenger, namely:

$\Delta C_{1,2} = C_1 - C_2$, ($where\ C_1 > C_2$), C_1 is considered the challenger

$\Delta B_{1,2} = B_1 - B_2$

$BCR_{1,2} = \frac{\Delta B}{\Delta C}$

If $BCR_{1,2}$>1, then, project 1 is the survivor and it moves on to challenge the other alternatives

If $BCR_{1,2}$<1, then project 2 is the survivor and it moves on to challenge the other alternatives

Then keep on comparing pairs with a challenger from the previous iteration until there is only one survivor left with BCR >1

Incremental analysis is a powerful tool for analyzing mutually exclusive alternatives. We will explore the technique in greater details in the follow up course.

Capitalized Cost-Capitalized Worth (CC or CW)

Capitalized Cost (CC) or Capitalized Worth (CW) is defined as the present worth (PW) equivalent of an infinitely long series of cashflows, that is the PW of an alternative that would last forever. It is the present worth of a constant annual cost over an infinite horizon. It is the amount of money that must be set aside now at some interest rate so as to have the money available to provide the service indefinitely. Capitalized worth is the present worth of a perpetuity. An infinite series of cash flows is unusual in a typical business environment, but it is quite common in governmental or public arena where projects are of long-term duration. The need for roads, bridges, railways, dams, irrigation, hospitals, police, tunnels, highways, pipelines, etc. is sometimes considered to be perpetual in which case the present worth of the costs associated with such projects would have an infinite analyses period ($n=\infty$). Another practical example of CW is the establishment of endowment funds (e.g., student scholarships or chaired professorships) where the estimated life is infinite.

Operationally, CC (or CW) is a special variation of PW and involves determining the PW of all revenues or expenses over an infinite length of time in perpetuity. In certain cases, if only expenses are considered, then the result of the approach is referred to as capitalized cost (CC). To capitalize is to record a cost or expense on the balance sheet for the purposes of delaying full recognition **of the expense**. In general, capitalizing expenses is beneficial as companies acquiring new assets with long-term lifespans can amortize or depreciate the costs. CW is a convenient basis for comparing mutually exclusive alternatives when the period needed for the service is infinitely long.

By definition: $PW = PV = A\left(P/_A, i\%, n\right) = A\left[\frac{(1+I)^n-1}{i(1+i)^n}\right] = \frac{A}{i}\left[\frac{(1+i)^n-1}{(1+i)^n}\right]$

$$\rightarrow\rightarrow PW = \frac{A}{i}\left[\frac{(1+i)^n-1}{(1+i)^n}\right] = \frac{A}{i}\left[\frac{(1+i)^n}{(1+i)^n} - \frac{1}{(1+i)^n}\right] = \frac{A}{i}\left[1 - \frac{1}{(1+i)^n}\right]$$

For CC (CW), $n \to \infty, PW = \frac{A}{i}\left[1 - \frac{1}{(1+i)^{n\to\infty}}\right] = \frac{A}{i}[1-0] = \left\|\frac{A}{i}\right\| = CC$

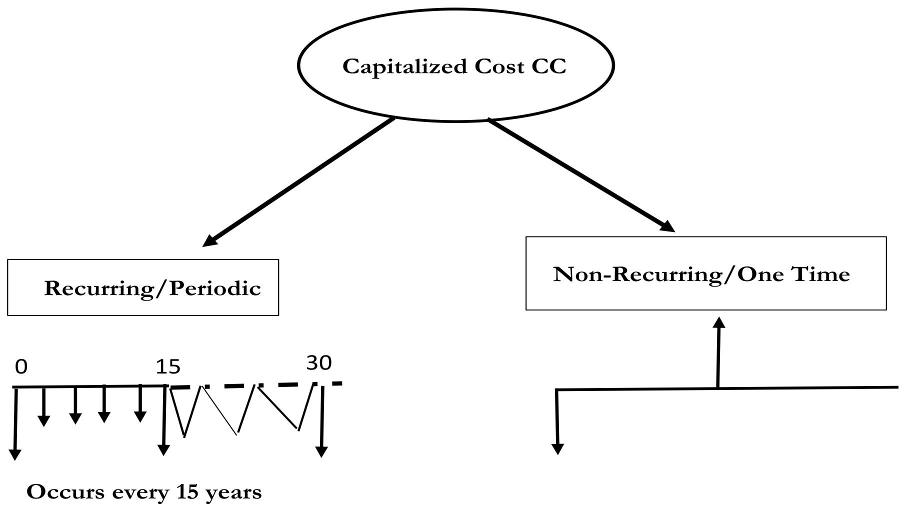

Figure 21. Capitalized Cost Framework

8.1 Approach to Implement CC(CW)

The approach to the computation of CC can be implemented in 5 distinct steps

1. Draw a cash flow diagram showing 2 cycles of recurring cash flows and any nonrecurring cash flows. (This initial cash flow diagram is quite instructive and perhaps more important when dealing with capital cost).
2. Find the PW of all nonrecurring amounts. This is the CC_1 value.
3. Find the Equivalent Uniform Annual Worth AW through one life cycle of all recurring amounts. This is the same value in all succeeding cycles. Add this to all other uniform amounts occurring in years one through infinity and the result is the total Equivalent Annual Worth (AW). This is sufficient to represent all other occurrences since the amount is the same for all succeeding years. That is $A_1, A_2, A_3, A_4, \ldots.$
4. Divide the AW in step 3 by the interest rate i to obtain a CC value, where $CC_2 = \frac{\sum AW}{i}$

5. Sum the CC values obtained in step 2 and 4 to give CC_T.

Example: Everglades county has recently acquired a software that would track electricity usage county wide as part of the county solar program. The county commissioners would like to know the total equivalent costs of the future cost of the software to determine the budgetary requirements since software would be used perpetually for an indefinite period. The estimated building cost is $350,000. The initial system cost including installation and computer equipment is $300,000. There is a recurring warranty cost of $20,000 every 8 years. There is also an equipment upgrade charge of $50,000 every 5 years. A system annual maintenance cost of $8,000 is assessed for the first 5 years, and $10,000 for subsequent years. Assume that the Average cost of capital (WACC) is 10%. If the system will be used indefinitely, a) find (the capitalized cost CC) now and b) for each year thereafter (AW).

Step 1: Sketch the Cash Flow Diagram (see figure 22)

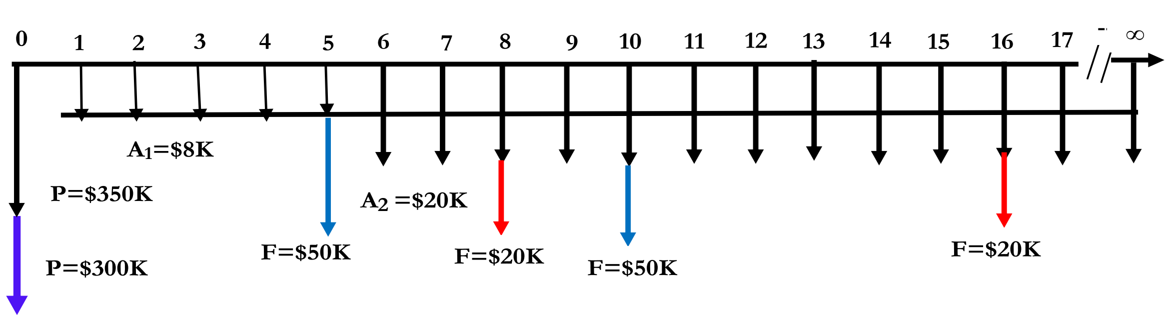

Figure 22: Cash Flow for Capitalized cost Example

Step 2: Find the PW of all non-recurring amounts.

Nonrecurring amounts are:

1. Building costs of $350K
2. System Cost of $300K

This is the CC value: $PW_1 = CC_1 = -\$350{,}000 + (-\$300{,}000\,)$

$PW_1 = CC_1 = -350{,}000 - \$300{,}000 = -\$650{,}000$

Step 3: AW of 1 cycle of recurring cash flows

1. Annual maintenance =$8k for 5 years
2. Annual maintenance =$10k from year 6 on
3. Warranty cost= $20K every 8 years
4. Equipment upgrades of $50K every 5 years

For #1 & 2, A system annual maintenance cost of $8,000 is assessed for the first 5 years, and $10,000 for subsequent years. There two possible ways to look at this:

i) -$8,000; 0 → ∞, and -$2,000, 6 → ∞, OR

ii) -$8,000; 0 → 5, and -$10,000, 6 → ∞, We will use alternative i) because it reflects better the true situation. If we do that, then year 5 now becomes time zero from the point of view of the Annual Worth (AW) of the subsequent occurrences of those annual payments of (\$10,000 less \$8,000=\$2,000) of -\$2,000, 6 → ∞ so there will be no discontinuity.

For #1: $-\$8{,}000;\ 0\ \rightarrow\ \infty, A_1 = -\$8K$

For #2: -\$2,000, 6 → ∞, we have $\frac{AW}{i} = \frac{-\$2000}{i},\ for\ 6\ to\ \infty$

Hence $PW = CC_{22} = \frac{-\$2000}{i}\left({}^{P}/_{F}, 10\%, 5\right) = -\$20{,}000(1.1)^{-5} = -\$12{,}418.43$ **(*Note: n=5, not 6*)**

For #3, $A_3 = -\$20K\left({}^{A}/_{F}, 10\%, 8\right) = -\$20{,}000\left[\frac{i}{(1+i)^n-1}\right] = -\$20{,}000\left[\frac{0.1}{(1.1)^8-1}\right] = -\$1{,}748.88$

For #4, $A_4 = -\$50K\left({}^{A}/_{F}, 10\%, 5\right) = -\$50{,}000\left[\frac{i}{(1+i)^n-1}\right] = -\$50{,}000\left[\frac{0.1}{(1.1)^5-1}\right] = -\$4{,}372.20$

Step 4: Divide the AW $(\boldsymbol{A_i})$in step 3 by the interest rate i to obtain a CC value

$$CC_{21} = \frac{A_1}{i} = \frac{-\$8{,}000}{0.1} = -\$80{,}000$$
$$CC_{22} = -\$12{,}418.43$$
$$CC_{23} = \frac{A_3}{i} = \frac{-\$1{,}748.88}{0.1} = -\$17{,}488.80$$
$$CC_{24} = \frac{A_4}{i} = \frac{-\$4{,}372.20}{0.1} = -\$43{,}722.00$$
$$CC_2 = CC_{21} + CC_{22} + CC_{23} + CC_{24} = -\$(80{,}000 + 12{,}418.43 + 17{,}488.80 + 43{,}722.00)$$
$$\boldsymbol{CC_2 = -\$153{,}629.23}$$

Step 5: Sum the CC values obtained in step 2 and 4 to give $\boldsymbol{CC_T}$.

$$\mathbf{CC_T = CC_1 + CC_2 = -\$650{,}000 - \$153{,}629.23 = -\$803{,}629.23}$$
$$\mathrm{CC} = \frac{\mathrm{A}}{\mathrm{i}} = \frac{\mathrm{AW}}{\mathrm{i}} \rightarrow \mathrm{AW} = \mathrm{CC(i)}$$
$$\boldsymbol{AW = CC_T(i) = (-\$803{,}629.23)(0.1) = -\$80{,}362.923}$$

Solution

a) $\boldsymbol{CC_T = -\$803{,}629.23}$, b) $\boldsymbol{AW = -\$80{,}362.923}$

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