A SunCam Online Continuing Education Course

Motors/Generators/NEC

Course I—Rotating DC Machinery

Course II—Rotating AC Machinery

Course III—NEC: Motors & Generators

Notational Methods/Revolving & Stationary Fields/Armature Reaction & Compensation
Series & Shunt Machines

by

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Nomenclature[1]

a	number of parallel armature paths	-
a	ratio of transformation	-
A	area	m^2
B	magnetic flux density	T
B	susceptance	S
B, $\mathbf{B}$	magnetic field	Wb/m (T·m)
E	generated emf	V
E	energy	J
E	generated voltage	V[2]
f	electrical frequency	Hz
F,$\boldsymbol{F}$	force	N
G	conductance	S
I,$\boldsymbol{I}$	constant or rms current	A
I_B	magnetization (quadrature current)	A
I_G	in-phase component of exciting current	A
k	constant	various[3]
l,$\mathbf{L}$	length	m
n	rotational speed	rev/min
N	number of turns/items	-
N	number of series armature paths	-
p	number of poles or poles/phase	-
P	power or power loss	W
pf	power factor	-
q	number of loops	-
R	resistance	Ω
r	radius	m
r, R	resistance	Ω
s	slip	-
S	apparent power	VA
SCR	short-circuit ratio	-

[1] Not all the nomenclature, symbols, or subscripts may be used in this course—but they are related and may be found when reviewing the references listed for further information. Further, all the nomenclature, symbols, or subscripts will be found in of many electrical courses (on SunCam, PDH Academy, and also in many texts). For guidance on nomenclature, symbols, and electrical graphics: IEEE 280-2021, IEEE Standard Letter Symbols for Quantities Used in Electrical Science and Electrical Engineering, New York: IEEE; and IEEE 315-1975, Graphic Symbols for Electrical and Electronics Diagrams, New York: IEEE, approved 1975, reaffirmed 1993.

[2] Generated voltages are traditionally represented as "E" as the symbol but often "V" is used in everyday contexts. The international standard symbol is "U". The usage varies with the text.

[3] Rotating machines have numerous constants with differing names and subscripts. Anytime a subscript or superscript changes on the symbology for the constant, some different term (or possibly units) has come into play. Constants include torque (or motor), back-emf (or electrical or voltage), electrical time, mechanical time, power. See Table 1.

SR	speed regulation	-
S_R	slew rate	V/s
SR	speed regulation	%
t	time	s
T	torque	N□m (ft-lbf)
T	period	s
v	variable voltage	V
V	constant or rms voltage	V
V	line voltage	V
v,**v**	velocity	m/s
V_0	generated voltage	V
VR	voltage regulation	%
X	inductance or reactance	Ω
Y	admittance	S
z	number of conductors	-
Z	impedance	Ω

Symbols

δ	torque angle	rad
η	efficiency	-
θ	angle or phase angle difference	rad
κ	torque conversion factor	-
ϕ	angle	rad
ϕ	flux	Wb
Φ	magnetic flux	Wb
ω	angular frequency	rad/s
ω_{mech}	rotational speed	rad/s

Subscripts

0	initial	-
0	stator	-
1	equivalent stator	-
2	equivalent rotor	-
a	armature	-
adj	adjusted	-
aux	auxiliary	-
ave	average	-
b	blocked rotor	-
c	capacitor	-
CEMF	counter-electromotive force	-
cp	commutating pole	-

Cu	copper	-
d	direct	-
ds	direct axis	-
E	emf	-
e	electrical, emf	-
eff	effective	-
f	field	-
f	final	-
fl	full load	-
g	generator	-
h	hysteresis	-
L	line, line-to-neutral, or load	-
m	mechanical or motor	-
max	maximum	-
mech	mechanical	-
net	net field	-
nl	no load	-
n	rotational speed/torque[4]	-
oc	open circuit	-
p	phase	-
pf	power factor	-
pu	per unit	-
q	quadrature	-
r	rotor	-
rev/min	revolutions/minute	-
s	synchronous	-
sc	short circuit	-
st	stator	-
sync	synchronizing	-
t	terminal, total, torque	-
T	torque	-
x	armature	-

[4] Related to the inertia of the rotor. Sometimes shown as "*in*" for inertia. The constant converts torque to speed and is related to a motor's resistance and voltage.

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COURSE REFERENCES

The theoretical information is primarily from one of the author's books, Ref. [A]. The NEC Ref. [B] is always a useful source for electrical engineers. Information useful in many aspects of electric engineering may be found in [C] and [D]. Reference [E] has detailed descriptions of analysis techniques. Reference [F] covers many terms in EE with excellent definitions and explanations. Reference [G] is one of the most comprehensive and best explained texts on motor and generator theory. The appendices (A-F) cover information useful in many engineering tasks with App. (G) providing a side by side comparison of electric and magnetic equations. Use these texts or their counterparts for in-depth information. References in bold are highly recommended.

This course will focus on basics, that rarely change, and provide the basis for all other knowledge.

Introduction

Electromagnetic devices, including sensors, relays, transformers, reactors (inductors), and rotating machines, operate using a similar set of electrical principles. These principles are applicable to each device, with each using a different application of the theory to accomplish a given task. Understanding these principles is important in understanding a wide variety of devices and machinery.

Notational Methods

Conventional current flow is used as a standard (i.e., the most widely accepted) in electrical engineering. *Current* is said to flow from positive terminals to negative terminals, while electrons flow in the opposite direction—from negative terminals to positive terminals. A *source* delivers power and a *load* absorbs power.

Figure 1 shows two black boxes, each with a positive and negative terminal. Current direction is as shown. The source and load boxes are as follows.

- When current flows out of the positive terminal, the circuit element is a source; therefore, A is the source in Fig. 1.
- When current flows into the positive terminal, the circuit element is a load; therefore, B is the load in Fig. 1.

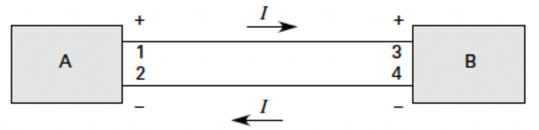


Figure 1: Source and Load

The sign notation for voltage uses *polarity markings*, as shown in Fig. 1. For example, from Fig. 1, E_{12} is the voltage between the terminals of source A with the voltage at 1 positive with respect to the voltage at 2. For load B, the same convention applies, where V_{34} indicates the voltage at 3 is positive with respect to the voltage at 4.

The *single-subscript notation* is similar, but uses one subscript and one polarity marking. When a reference node is used, the single-subscript notation can be called the *potential level representation*. The potential level representation is useful in understanding electronic circuit operation used to control and monitor motors and generators. For example, if the line between terminals 2 and 4 in Fig. 1 is considered the reference, then E_{12} can be represented as E_1. Similarly, V_{34} can be represented as V_3. Using Kirchhoff's voltage law (KVL) gives

Equation 1: Notational Representation

$$E_{12} - V_{34} = E_1 - V_3 = 0$$

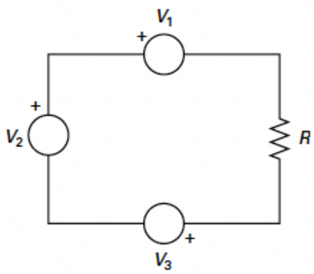
Equation 1 assumes no voltage drop over the lines between the source and the load. This assumption may be used for the internals of motors and generators and short distribution systems, but cannot be used if higher accuracy is required or if excessive voltage drop will result in improper operation of the equipment.

The voltage and currents can be either AC or DC quantities; the representation remains the same. For AC quantities, the polarities, subscripts, and current directions represent an instant in time. Though the polarity and current directions change, the relationship remains, and what was a positive voltage from 1 to 2 becomes a negative voltage from 1 to 2 during the opposite halfcycle.

Example 1

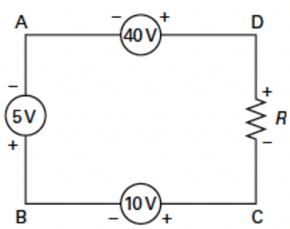
In the following circuit, the voltages use the single subscript notation and have the following values. What is the actual polarity of the voltages?

$V_1 = -40$ V
$V_2 = -5$ V
$V_3 = +10$ V



Solution

The voltages V_1 and V_2 are both given as negative. Therefore, the positive sign is on the terminal opposite. The result is as follows.

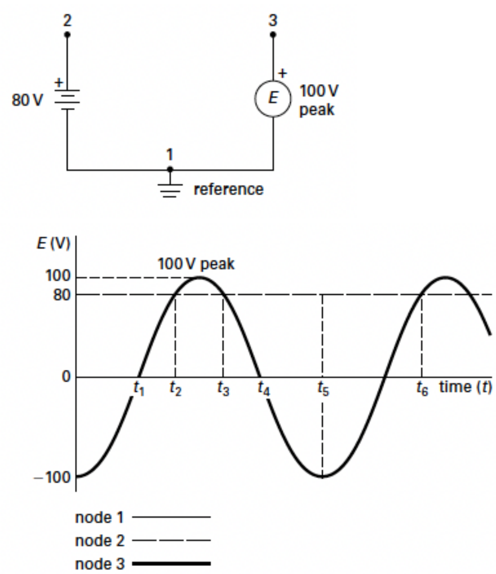


The potential across resistor R is 25 V with the polarity as shown. While it can appear that the potentials at nodes B and C have two values and that potential is 30 V, this is not the case. With single-subscript notation, the potentials are neither inherently positive nor negative. Instead, they have a value in reference to, or with respect to, another location.

The key is to know that the negative sign indicates the potential is the opposite of that shown AND, to always use either the input or output polarity in KVL as the reference, OR for KCL use either input as positive or output as positive. By remaining consistent, the resulting equations will give the correct value.

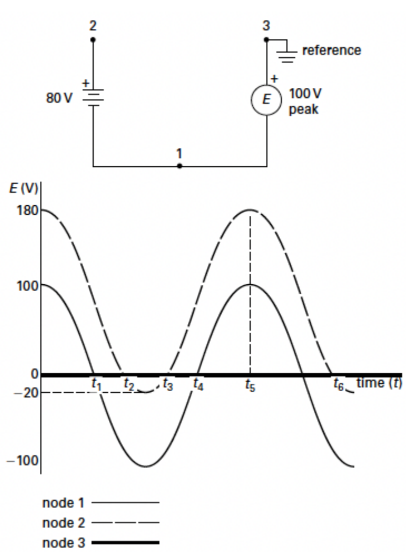
Example 2

Consider the circuit shown with the reference level as indicated. The potentials are graphed with respect to the reference node at 1. What does the graph look like with the reference node at node 3?



Solution

The graph is as follows.



Faraday's Law

Electric rotating machines and electromechanical devices (such as relays) consist of coupled electric and magnetic circuits. An electric circuit provides a path for the flow of electric current, and a magnetic circuit provides a path for magnetic flux. Sources of magnetic flux include permanent magnets and electric currents.

Faraday's law of electromagnetic induction describes how these coupled circuits interact. It states that the electromotive force (emf) induced in a circuit by a changing magnetic field is equal to the negative rate of change of the flux linking the circuit.

Equation 2: Faraday's Law

$$v = -\frac{d\phi}{dt}$$

The law refers to the rate of change for flux linking the circuit. Such flux linkage is the product of the number of turns, *N*, in a coil and the magnetic flux passing through the coil.

Equation 3: Faraday's Law, Turns

$$v = -N\frac{d\phi}{dt}$$

The rate of change can originate by two means: the flux can change with time, as indicated in Eq. 2 and Eq. 3, or the flux can remain constant while the conductor moves with velocity *v* over a path, *l*, to "cut" the lines of flux, giving Eq. 4.[5] The negative sign is not necessary since the mixed triple product provides the correct direction.

Equation 4: Faraday's Law, Rate of Change Options

$$v = \oint \left(\mathbf{v} \times \mathbf{B}\right) \cdot d\mathbf{l}$$

In simpler terms, Faraday's law states that if a flux linking a loop (or turn) varies as a function of time, a voltage is induced between the terminals that is proportional to the rate of change of flux. The negative sign is often excluded since the direction is clear or necessary only for the original

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[5] This sentence can be used as a thumb-rule, or memory aid, for determining if one is dealing with a motor or generator. That is a "current-carrying conductor in a magnetic field" is a motor. And, a "conductor moving in a magnetic field" is a generator. In the first, one injects the current. In the second, one injects the motion via mechanical means.

designer of the equipment. Equation 5 is used to make approximate calculations for fluxes that change with time.

Equation 5: Faraday's Law, Approximate

$$v = N\frac{\Delta\phi}{\Delta t}$$

In rotating machines, the angle is often a right angle (or close to one).[6] If the conductor is moving at constant speed at right angles to the magnetic field, the voltage induced is given by Eq. 6.

Equation 6: Faraday's Law, Right Angles

$$V = Blv$$

B is the magnetic flux density. l is the length of the conductor in the magnetic field. The symbol v is the velocity of the conductor through the magnetic field. If the conductor and the magnetic field are not at right angles, a sine term may be added to account for the angle between the magnetic field and the conductor.

Equation 7: Faraday's Law, Non-Right Angles

$$V = Blv\sin\theta$$

For motor action, the force is the quantity of first use. To obtain the force equivalent of Eq. 7, the velocity of the conductor, v, is replaced by the velocity of the positive charges, represented by the current, **I**.[7] This transition is accomplished by Eq. 8. The vector **l**, representing a unit vector in the direction of the current **I**, is often used in the place of the current vector **I**.[8]

Equation 8: Motor Force, Current

$$\mathbf{F} = \oint \mathbf{I}\times\mathbf{B}\,dl = \oint I\,d\mathbf{l}\times\mathbf{B} = I\mathbf{L}\times\mathbf{B}$$

In terms of individual positive charges, Eq. 8 becomes

[6] The direction of the potential can be found from considering the direction of $\mathbf{v} \times \mathbf{B}$.
[7] The use of unit analysis can aid in confirming the validity of such a substitution.
[8] The vector **L** in equation Eq. 32.8 represents the total length of the conductor carrying current in the magnetic field. It is capitalized in this case only to avoid confusion with the current variable I. Where magnitudes and the italic l are used, equations are not prone to the same confusion, and the lowercase is more prominent.

Equation 9: Motor Force, Positive Charges

$$\mathbf{F} = I\mathbf{L} \times \mathbf{B} = Q\mathbf{v} \times \mathbf{B}$$

The same substitution (i.e., I for v) made in Eq. 6 and Eq. 7 results in the following force equations (for generator action). Equation 10 determines the force felt on a conductor with length, l, at right angles to a magnetic field of flux density, **B**, carrying a current I. If the current (i.e., the conductor) is not at right angles to the magnetic field, then Eq. 11 must be used.

Equation 10: Generator Force, Current

$$F = BlI$$

Equation 11: Generator Force, Non-Right Angles

$$F = BlI \sin\theta$$

Equation 1 through Eq. 11 may be used for calculations on either AC or DC motors or for motor action in generators.

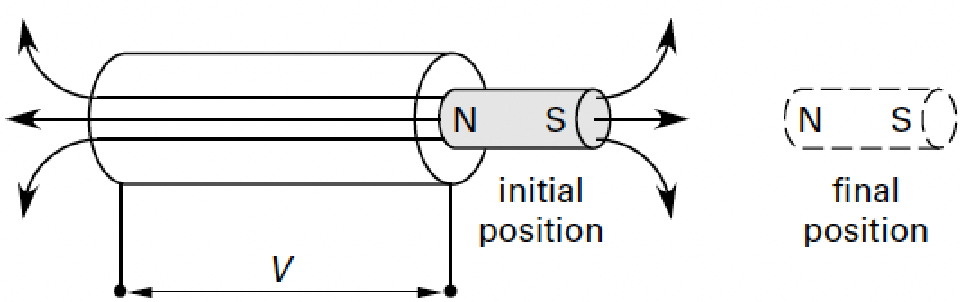
NOTE

The equations labeled Motor Force represent the force output resulting from the current and the resulting relative motion. The equations labeled Generator Force represent the force input required to generate the relative motion resulting in the current.

Example 3

A coil of wire with 5000 turns surrounds a permanent magnet. The magnet is rapidly withdrawn, which causes the flux in the coil to change from 5×10^{-3} Wb to 1×10^{-3} Wb in 1×10^{-3} s.

What voltage is induced between the terminals?



Solution

The potential can be calculated using Eq.5.

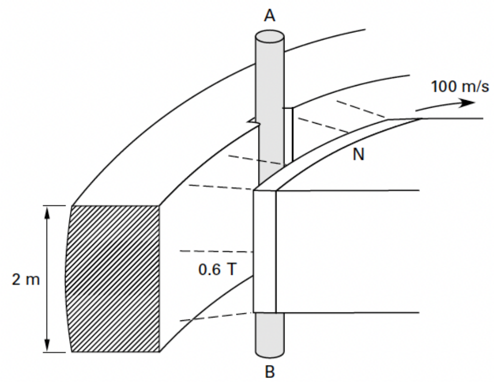
$$v = N\frac{\Delta\phi}{\Delta t} = (5000)\left(\frac{5\times10^{-3}\ \text{Wb} - 1\times10^{-3}\ \text{Wb}}{1\times10^{-3}\ \text{s}}\right) = 20{,}000\ \text{V}$$

Fireplace igniters, barbecue pit lighters, and other automatic lighters operate on the principles illustrated in Ex. 3. They use springs to initiate the physical movement of the magnet, rapidly changing the flux. Ignition coils use the same principal, but use a switch that removes the current quickly and collapses the magnetic field, thereby inducing the rapid change.

Example 4

A large generator contains conductors that have 2 m of their length (A to B) on one side within a 0.6 T magnetic field, as shown. The rotating magnetic field moves at 100 m/s.

What is the voltage induced in the conductor between points A and B?



Solution

The induced voltage can be calculated from Eq. 6.

$$V = Blv = (0.6\ \text{T})(2\ \text{m})\left(100\ \frac{\text{m}}{\text{s}}\right) = 120\ \text{V}$$

Revolving Field Rotating Machine

Consider a stationary metal ring inside of which is located a rotating magnet, as shown in Fig. 2. The rotating magnet may be a permanent magnet or an electromagnet.

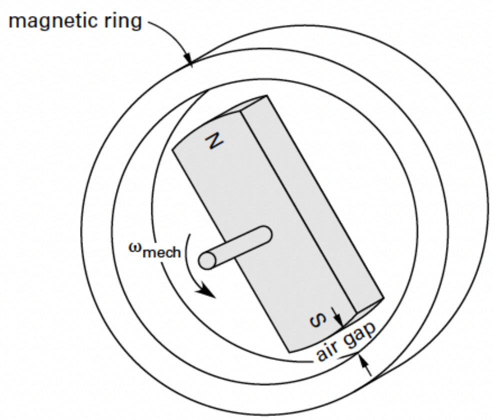


Figure 2: Elementary Machine with Rotating Field

The stationary metallic ring is meant to reduce the reluctance of the magnetic circuit. The magnetic circuit runs from the north pole, across the air gap, into the ring, around the ring (in both directions), across the air gap, into the south pole, and then internally back to the north pole.

A single turn of a rectangular conductor is embedded in the ring as shown in Fig. 3(a). The coil is physically attached to the ring and does not rotate. With the revolving magnet included, the illustration becomes that of Fig. 3(b). When the revolving magnetic field is in the plane of the rectangular conductor, the induced voltage is at a maximum with the polarity as shown. (Consider this the 0° position.) Explained differently, when the poles are closest to conductor segments AB and CD, the voltage is at a maximum.

With the revolving magnet perpendicular to the plane of the rectangular conductor, the induced voltage is at a minimum (zero) with the polarity as shown in Fig. 3(c). (This is the 270° position.) If the rotation is 60 rev/min, the voltage between terminals A and D is as shown in Fig. 3(d) when considered in reference to the angle of rotation. When considered in reference to time, the voltage is as shown in Fig. 3(e). The advantage of the revolving field is that the electromagnet carries a much lower current through revolving connections and permanent, nonmoving connections can be made to the coil and field. *Nearly all AC generators are of the revolving-field type.*[9]

[9] This setup can be an AC generator or motor. To act as a motor, a rotating field would need to be established, and the rotor could be shorted or the resistance (now shown as the load) could act to control the current in the rotor, and thereby the speed. Physically, AC generator construction and AC motor construction differ significantly from one another, unlike construction of DC generators and motors, which can be identical.

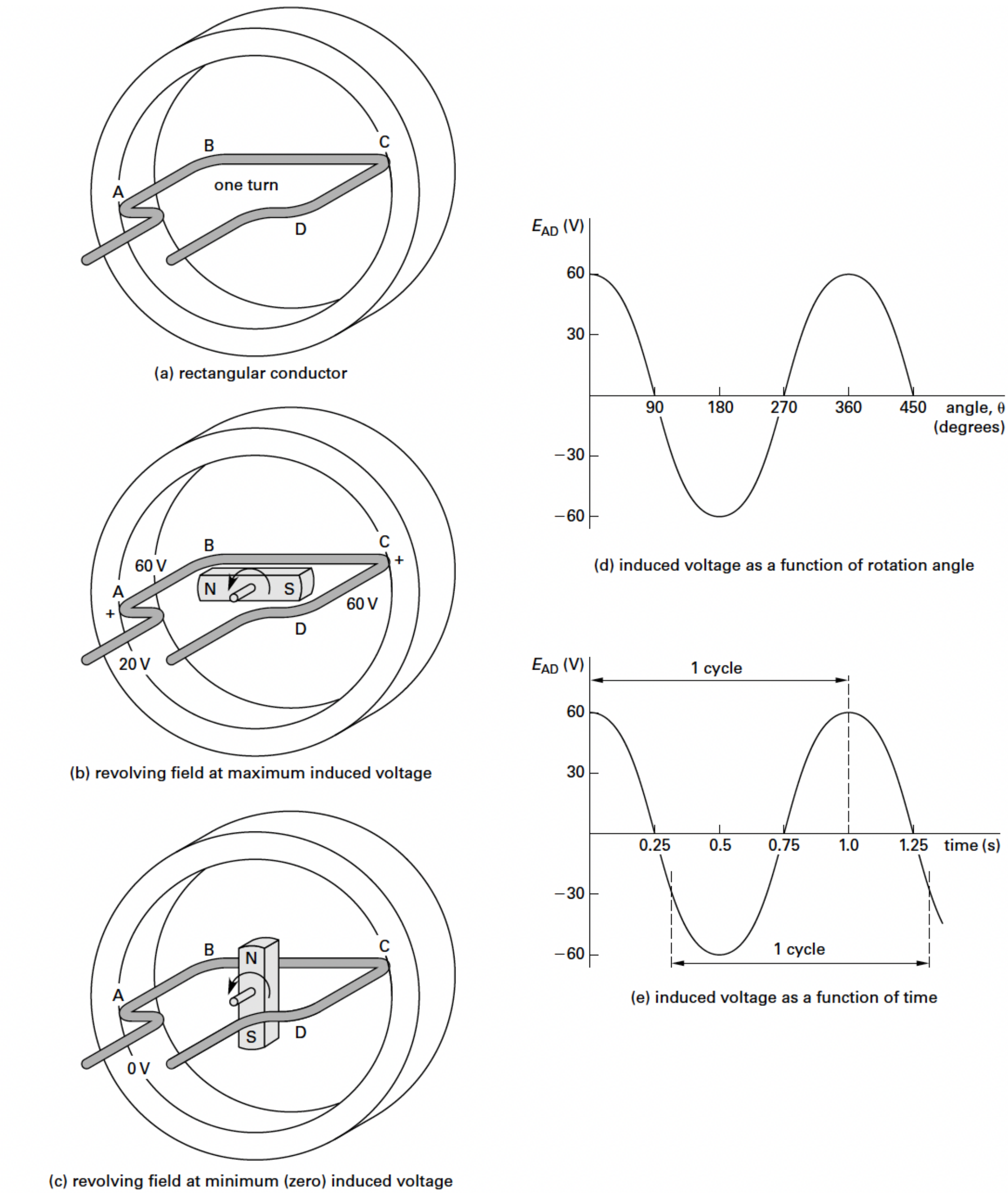


Figure 3: Revolving Field

Stationary Field Rotating Machine

Consider a stationary field located around a moving coil, as in Fig. 4. The stationary field may be a permanent magnet or an electromagnet.

Just as with the rotating field, the magnitude of the induced voltage, the polarity of the voltage, and the graph of the voltage as a function of angle or time all remain the same (assuming the same rotational speed). The principles and equations are identical. However, the difference is that within a stationary field, a direct connection cannot be made to the coil.

Two basic options exist for the connection to points A and D in Fig. 4. The first is a connection made with slip rings joined to each end of the coil and mounted on a shaft, but insulated from the shaft, as shown in Fig. 5. With this connection, when the coil is at the position shown in Fig. 4, the polarity of brush X is positive and brush Y is negative. When the coil segment AB of Fig. 4 has rotated such that it is directly under the south pole (i.e., the position where segment CD is shown), the polarity reverses. The output is identical to that shown in Fig. 3(d) and Fig. 3(e).

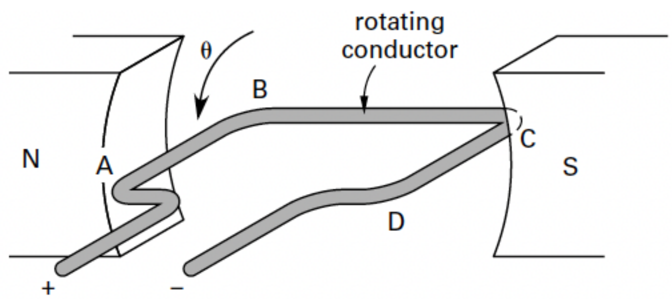


Figure 4: Elementary Machine with Stationary Field

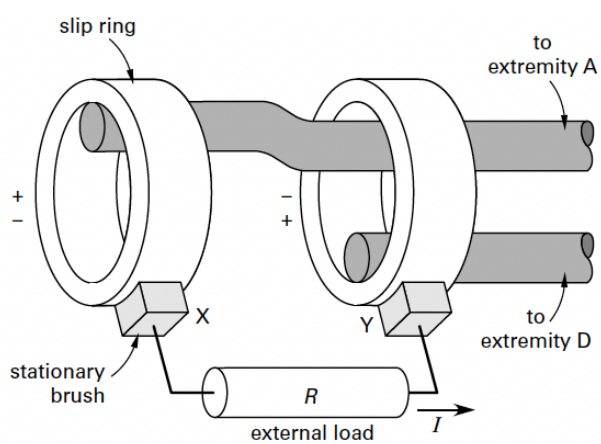
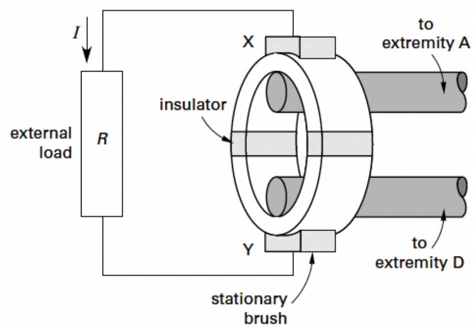


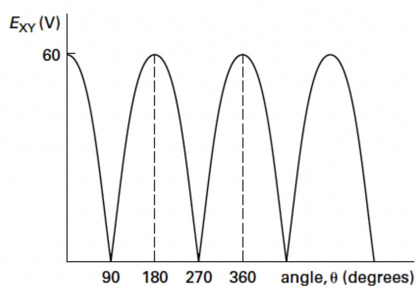
Figure 5: Slip Ring and Brushes Connection

The second option for the connection is a commutator, as shown in Fig. 6. In its simplest form, it comprises a slip ring cut in half with an electrical insulator between the two halves. One segment is connected to coil end A and the other to coil end D of Fig. 4, as illustrated in Fig. 6(a). The commutator revolves with the coil, so brush X is always connected to the segment of the coil under the north pole, which means it is always positive. The commutator rectifies the AC output to DC, as shown in Fig. 6(b). A commutator serves dual functions—it provides the direct connection point between the rotating portion of the machine (the rotor) to the stationary portion (the stator), and it reverses current flow mechanically.

The advantage of the stationary field is that rectification can take place on the rotating portion using a mechanical connection rather than power electronic circuitry. *Nearly all DC generators are of the stationary-field type*. Additionally, with this setup, the machine can be a DC generator (if the input is a mechanical torque on the rotor) or a DC motor (if the input is a DC current on the rotor).[10]



(a) commutator



(b) commutator DC output

Figure 6: Commutator Connection

[10] To make the machine a generator, mechanical power must be applied to the rotor, resulting in relative motion between the stator field and the rotor armature. To make the machine a motor, a current is applied to the rotor, resulting in the interaction of two magnetic fields, which then causes motion. The construction of the machine stays the same.

DC Waveshape Improvement

A *generator* is a device that produces DC potential. The actual voltage induced is sinusoidal (i.e., AC). However, brushes on *split-ring commutators* make the connection to the rotating armature and rectify the AC potential.[11] The more coils there are, the smoother the DC voltage. Two commutator segments are needed for each coil, and each coil produces its own sine wave. Because mechanical commutation is difficult unless the emf is produced in a rotating armature, DC generators are based on the simple design shown in Fig. 32.7.

DC generators suffer from several limitations. The rotating coils must be well insulated to prevent shorting with the high voltages that are induced. Structural bracing is required to counteract the large centrifugal force that results from rotating many coils of wire. It is difficult to make efficient high-voltage, high-power connections through slip rings.

The armature in a simple DC generator consists of a single coil with several turns (loops) of wire. The two ends of the coil terminate at the commutator, which consists of a single ring split into two halves known as segments as is shown in Fig. 7. The brushes slide on the commutator and make contact with the adjacent segment every half-rotation of the coil. This produces a rectified (though not constant) potential, as shown in Fig. 8.

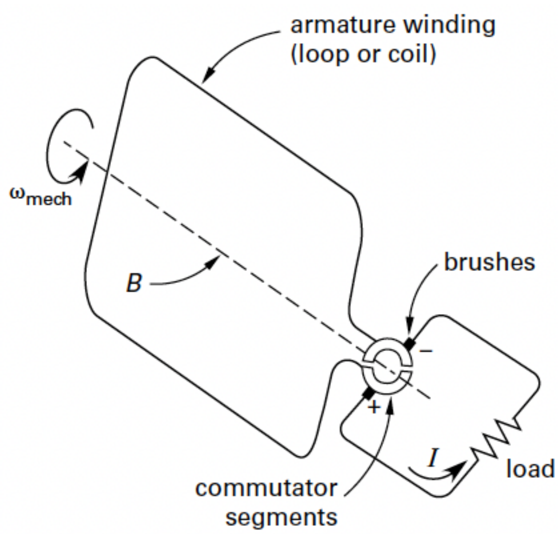


Figure 7: Commutator Action

[11] *Sparking* is one of the problems associated with commutators and occurs at points of low resistance. Brush resistance is nonlinear and drops as current increases.

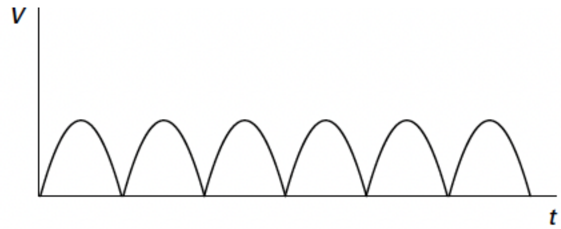


Figure 8: Rectified DC Voltage Induced in a Single Cell

Modern DC generators contain multiple coils connected in series, an arrangement known as closed-coil winding. See Fig. 9 for a two-coil, four-segment closed-coil armature. The coils are spaced uniformly around the armature core. The single-ring commutator is divided into as many pairs of segments as there are coils. There are only two brushes, however, located on opposite sides of the commutator. Because there are many coils, as the armature rotates, the brushes always make contact with two segments of the commutator that are in nearly the same positions relative to the magnetic field.

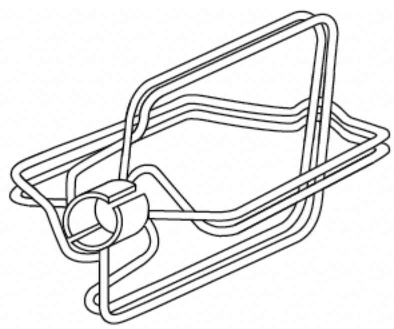


Figure 9: Two-Coil, Four-Segment Closed-Coil Armature

The average induced emf, $\overline{E}$, for a finite number of coils approaches the voltage induced with an infinite number of coils. N is the total number of series turns in all of the coils.

Equation 12: Induced EMF

$$\overline{E} \approx E_{\infty} = 2\left(\frac{n}{60}\right)NAB$$

Because the coils are connected in series (in the modern closed-coil winding arrangement), the emf induced is the sum of the emfs induced in the individual coils. The voltage induced in each coil of a DC generator with multiple coils is still sinusoidal, but the terminal output is nearly constant, and not a (rectified) sinusoid. (See Fig. 10.) The slight variations in the voltage are known as ripple. Because the output is nearly constant, the concept of electrical frequency has no meaning, and distinctions between maximum, effective, and average voltages are not made.

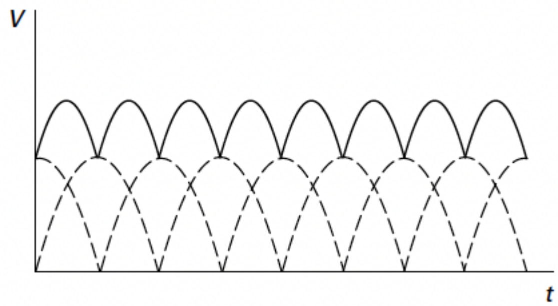


Figure 10: Rectified DC Voltage from a Two-Coil, Four-Segment Generator

It is also possible to use an open-coil connection (see Fig. 11), though this is seldom done. Lower voltages are produced, as the average induced voltage is the maximum voltage from a single coil.

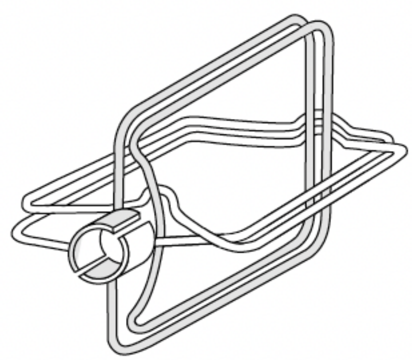


Figure 11: Two-Coil, Four-Segment Open-Coil Armature

The average emf, E, for a DC machine (motor or generator) is given by Eq. 13. Φ is the flux per pole. For a generator, the emf is greater than the armature voltage ($E > V_a$). For a motor, the armature voltage is greater than the back emf ($V_a > E$).

Equation 13: Average EMF

$$E = \frac{Np\Phi n}{60} = \frac{zp\Phi n}{60a} = \kappa_E \Phi n$$

Equation 14: EMF Constant

$$\kappa_E = \frac{zp}{60a}$$

Numerous constants, with varying symbology and units, exist. See Table 1 for examples along with definitions.

Table 1: Constants, Optional Symbology, Definitions

Electrical time constant	τ_e	s	The time for the motor current to reach 63% of its final value. Equal to L/R.
Torque constant	k_t	Nm/A	Also called the motor constant.
Electrical constant	k_e	Vs/rad	Same numerical value as the torque constant (in SI units). Also called voltage or back-emf constant.
Speed constant	k_s	rad/(Vs)	Inverse of electrical constant.
Mechanical time constant	τ_m	s	The time for the motor to go from rest to 63% of its final speed under constant voltage.
Rotor inertia	J	kgm^2	Often given in units gcm^2.

Of note, in the SI system the Torque Constant [or motor constant] and the Electrical Constant [or back-emf constant] are numerically equal. In US Customary Units the differ by a conversion factor between radian/second and revolutions/minute.[12]

More complex generators use additional coils and commutator segments, rather than the closed-coil arrangement, as shown in Fig. 12(a).[13] This not only more closely approximates a steady DC [see Fig. 12(b)], it smooths the opposing torque (from motor action) and lowers the vibration level. Additionally, it allows lower current levels per winding while maintaining the output, overcoming some of the limitations mentioned.

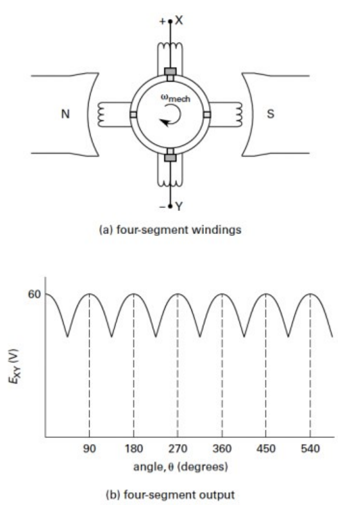


Figure 12: Four-Segment Commutation

[12] That is 60/2π ≈ 9.55.
[13] The waveform can also be improved by using electronic circuitry to filter the signal.

Armature Reaction

The field windings (either permanent or electromagnetic) of a generator in an AC or DC machine induce a voltage in the armature windings. The induced voltage creates current flow in the armature that results in a separate magnetic field. This armature field interacts with the original field to generate an armature reaction.[14]

Consider the magnetic field of the machine (generator) shown in Fig. 13. The dashed line represents the neutral plane. The magnetic field of the field windings (or permanent magnet) points from the north to the south pole, as shown in Fig. 32.13(a). Th*e neutral plane* is the location perpendicular to this field where windings on the armature (rotor) move parallel to the lines of force, so no voltage is induced.

In Fig. 13(b) the magnetic field of the armature (rotor) is shown. For a generator, the current flows as indicated. Use the cross product of the velocity, **v**, and the magnetic field, $\mathbf{B}_a$, of the armature to determine the direction of the current flow in the armature windings. The right-hand rule can be used to determine the direction of the armature field.

Each field interacts leaving only one net field, $\mathbf{B}_{\text{net}}$. This net field is the vector sum of the two interacting fields. Figure 13(c) shows the direction. Because of this shift in the direction of the net magnetic field, the neutral plane, which is perpendicular to it, also shifts. *In the case of a generator, the neutral plane shifts in the direction of rotation. For a motor, the neutral plane shifts opposite the direction of rotation.*

Consider the windings in locations X and Y and compare Fig. 12(a) to Fig. 13(b). Windings shorted by the brushes at terminals X and Y had no induced voltage present in the condition of Fig. 12(a). This is the reason for the selected physical location of the brushes. Current is flowing via terminals X and Y even though no voltage exists across the winding associated with the shorted commutator segments. When armature reaction occurs, the neutral plane shifts, and the shorted windings now have an induced voltage as indicated in the condition of Fig. 13(c). The result is a voltage difference across the shorted commutator segments resulting in arcing across the insulating segment between the shorted segments, directly under the brushes. The voltage difference causes

[14] An armature reaction occurs in both AC and DC machines. The impact in DC machines is dramatic. The arcing of brushes, no longer in the neutral plane (where they would be shorting commutator segments with zero induced voltage across them), can result in immediate damage to the machine and can be seen with the naked eye. In AC machines, the interaction results in non-sinusoidal components and time harmonics in the output. The physical distribution of the conductors results in space harmonics, which are not discussed here, though the method of analysis depends heavily on Fourier series in both time and space.

inefficient operation, noise, harmonics, and physical damage to the brushes and commutator segments.[15]

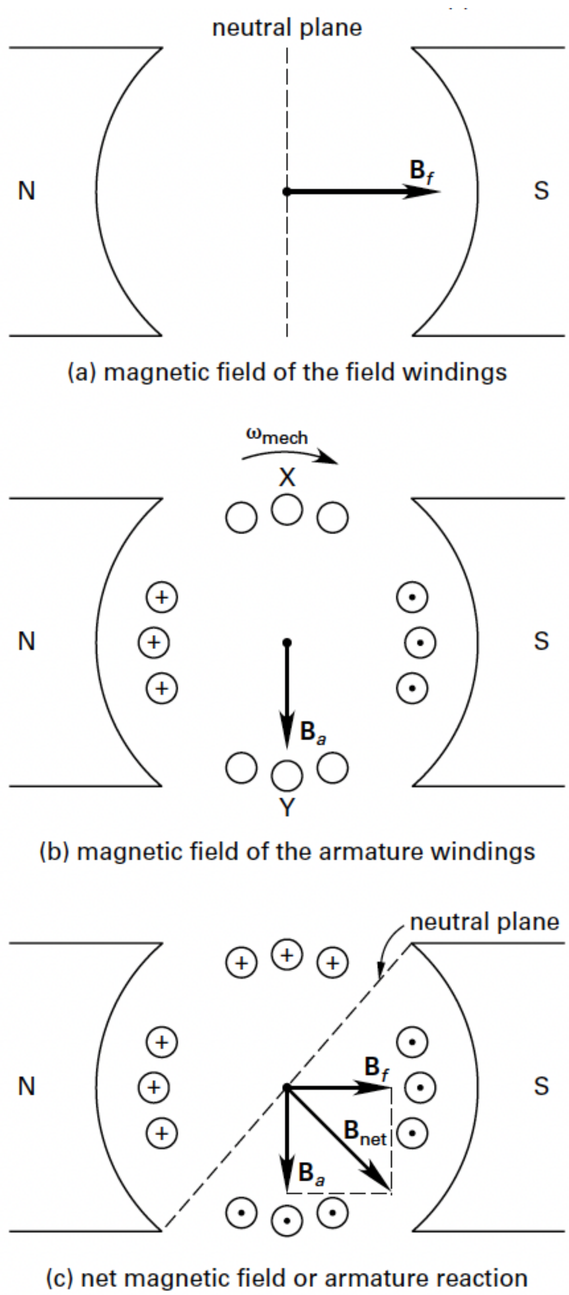


Figure 13: Armature Reaction Fields: Vector Approach

[15] It will not be difficult to spot when the brushes are NOT in the neutral plant. The result is quite the light show!

Figure 14 shows a closer view of armature reaction. In Fig. 14(a), the stationary magnetic field, $\mathbf{B}_f$, is shown for a generator under no load. In Fig. 14(b), the armature (rotor) magnetic field, $\mathbf{B}_a$, is shown.[16] In Fig. 14(c), armature reaction $\mathbf{B}_{net}$ is shown. The twisting of the magnetic field results in the associated movement of the neutral plane.

In addition to the problem of the neutral plane shift, the pole tip at A in Fig. 14(c), can become saturated. This causes the flux increase on the right side of the pole piece to be less than the decrease on the left side. As a result, the total flux, Φ_3, is less than the flux at no load, Φ_1. To prevent this potential instability, a series field (in series with the armature though wrapped on the stator) of one or two turns can be added to strengthen the shunt field (the field on the stator). This is called a *stabilized shunt*.

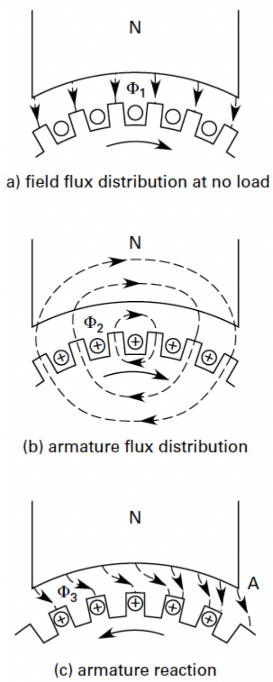


Figure 14: Armature Reaction Fields: Flux Approach

[16] Consider the direction of rotation in Fig. 13(b). The direction can be determined using the cross product of $\mathbf{v} \times \mathbf{B}$, with $\mathbf{v}$ being the velocity of the positive charges. Alternately, the right-hand rule can be used. Using the right-hand rule, the thumb is pointed in the direction of current flow on the rotor and the fingers curl in the direction of the surrounding magnetic field. For the conductors under the north pole, the magnetic field on top of the conductors is in the opposite direction of the north-south flux. The field on the bottom of the conductor is in the same direction as the north-south flux. This results in a weakening of the field above the conductors and a strengthening of the field below the conductors. The effect is a movement "up" or clockwise. If the magnetic field is mapped, it appears similar to a rubber band that has been stretched and is attempting to straighten out. This goes by the colloquial name *the rubber band theory of motor action*.

Commutating Poles

A way to correct for armature reaction is with the physical movement of the brushes to the new neutral plane. The obvious drawback to this correction is constructing such a movable device. Additionally, as the loading changes, the net magnetic field changes. The more frequent the loading changes, the more frequent the brush position movement. A more efficient approach is to use commutating poles.

Consider the individual magnetic fields of Fig. 15(a) and Fig. 15(b). It is the effect of the armature field in Fig. 15(b) that must be eliminated to prevent armature reaction. If a winding is placed in series with the armature, a commutating field, $\mathbf{B}_{cp}$, results, as shown in Fig. 15(c). This commutating field is in direct opposition to the armature field, as shown in Fig. 15(d). The commutating pole field cancels the shift due to the armature magnetic field, and the net result is the original magnetic field of Fig. 15(a). The neutral plane does not shift as it did in Fig. 13(c).

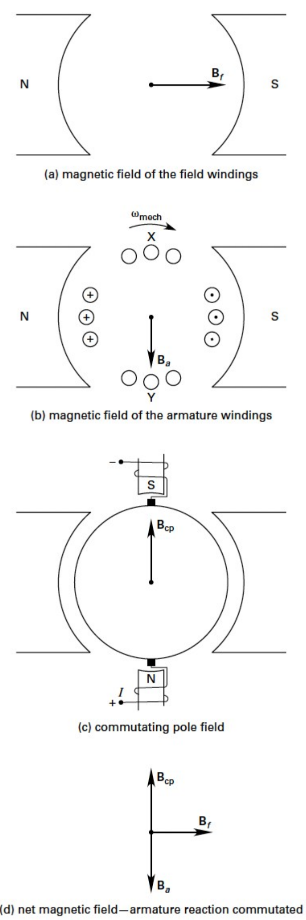
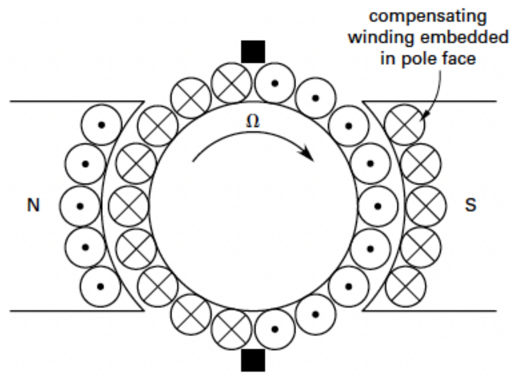


Figure 15: Commutating Poles

Compensating Windings

Large DC motors (i.e., those in the MW range) that operate with sudden acceleration, deceleration, or reversals can have large changes in armature reaction over a period of a few seconds. In such severe-duty cases, commutating poles and stabilizing windings are not enough to neutralize the armature magnetic field (i.e., the magnetomotive force, or mmf). In these cases, compensating windings are connected in series with the armature and distributed in slots cut into the pole faces of the main windings. (See Fig. 16.)



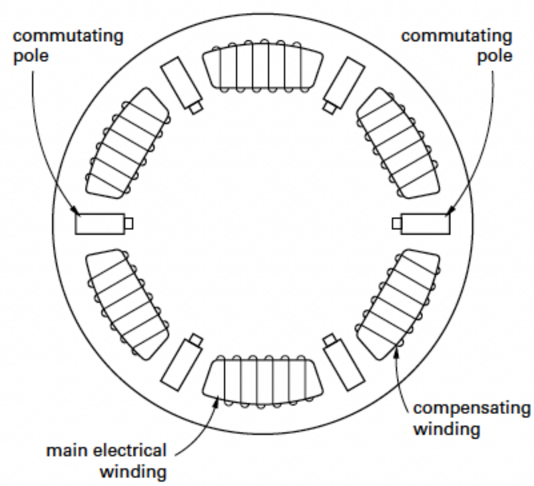


Figure 16: Six-Pole DC Motor with Commutating Poles and Compensating Windings

Generator and Motor Action

Recall that Eq. 6 (*V* = *Blv)* voltage. This equation represents the three items required for generator action; that is, for inducing a voltage.

• a magnetic field
• a conductor
• relative motion between the two

The force generated was given by Eq. 10 ($F = BlI$). This equation represents the three items required for motor action—that is, for the generation of a force (or for a torque when the force is applied across a moment arm, such as the radius of the rotor).

• a magnetic field
• a conductor
• current in the conductor

In a generator, a torque is applied to the shaft causing a relative motion between the conductors in the stator and the rotor, which carries the magnetic field. This is called *generator action*. Once there is relative motion, a voltage is induced in the stator windings resulting in current flow and the generation of a separate magnetic field that interacts with the first. The current flow is the desired output.

The interaction of magnetic fields can be understood in terms of *Lenz's law* (the current induced by an emf will always oppose the change that caused the emf). It can also be understood by noting that once the induced voltage causes a current flow, the conditions for motor action are met. That is, a current carrying conductor exists in a magnetic field. A force or torque is generated that opposes the motion of the rotor.

In a generator, generator action (the desired result) results in motor action (the undesired result). The motor action is a result of the conservation of energy. A constant input force or torque is required to continue generating an electrical output.

For a motor, the situation reverses. Motor action occurs first (the desired result), then generator action occurs (the undesirable output).

In any rotating machine there is generator action and motor action. If the generator action is the desired output, the opposing force or torque is called counter torque. If motor action is the desired output, the opposing voltage or current is called *counter-electromotive force* (or CEMF).

Rotating Machines: Overview

Rotating machines are broadly categorized as AC and DC machines. Both categories include machines that use power (i.e., motors) and those that generate power (alternators and generators).

Torque and Power

Torque and power are operating parameters. It takes power to turn an alternator or generator. A motor converts electrical power into mechanical power. In the SI system, power is given in kilowatts (kW). One horsepower is equivalent to 745.7 W. The relationship between torque and power is

Equation 15: Torque US System

$$T_{\text{ft-lbf}} = \frac{5252\left(P_{\text{horsepower}}\right)}{n_{\text{rev/min}}}$$

Equation 16: Torque SI System

$$T_{\text{N·m}} = \frac{9549\left(P_{\text{kW}}\right)}{n_{\text{rev/min}}} = \frac{1000\left(P_{\text{kW}}\right)}{\omega_{\text{mech}}}$$

There are many important torque parameters for motors. The *starting torque* (also known as *static torque*, *breakaway torque*, and *locked-rotor torque*) is the turning effort exerted in starting a load from rest. *Pull-up torque* (*acceleration torque*) is the minimum torque developed during the period of acceleration from rest to full speed. The *steady-state torque* must be provided to the load on a continuous basis. It establishes the temperature increase that the motor can withstand without deterioration. The *rated torque* is developed at rated speed and rated horsepower. Breakdown torque is the maximum torque the motor can develop without stalling (i.e., coming rapidly to a complete stop).

Equation 17 is the general torque expression for a rotating machine with N coils of cross-sectional area A, each carrying current I through a magnetic field of strength B.

Equation 17: General Torque Equation

$$T = NBAI\cos\omega t$$

Service Factor

The horsepower and torque ratings listed on the nameplate of a motor can be provided on a continuous basis without overheating. Motors can be operated at slightly higher loads without exceeding a safe temperature rise, but the higher temperature has a deteriorating effect on the winding insulation. (A general rule of thumb is that a motor loses two or three hours of useful life for each hour run at the factored load.) The ratio of the safe to standard loads is the service factor, usually expressed as a decimal. Service factors vary from 1.15 to 1.4, with the lower values going to larger, more efficient motors.

Equation 18: Service Factor

$$\text{service factor} = \frac{\text{safe load}}{\text{nameplate load}}$$

Motor Classification

The National Electrical Manufacturers Association (NEMA) has categorized motors in several ways: *speed classification* (constant-, adjustable-, multi-, varying-speed, etc.), *service classification* (general, definite, and special purpose), and motor class. *Motor class* is a primary indicator of the maximum motor operating temperature, which, in turn, depends on the type of insulation used on the conductors. The classes are Class A, 105°C; Class B, 130°C; Class F, 155°C; and Class H, 180°C.

Power Losses

The losses for all rotating machines can be divided into four categories.[17] *Copper losses*, P_{Cu}, are real power losses caused by wire and winding resistance. In a DC machine, copper losses occur in the armature and field windings as well as from the brush contact resistance. In an AC machine, copper losses occur in the armature and exciter field windings. There are no brush losses in an induction machine.

Equation 19: Copper Losses

$$P_{\text{Cu}} = \square\ I^2 R$$

Core losses, including hysteresis and eddy current losses, are constant losses that are independent of the load and, for that reason, are also known as *open-circuit* and *no-load losses*. In DC and

[17] The subject of transformer losses is an equivalent concept.

synchronous AC machines, core losses occur in the armature iron. In induction machines, core losses occur in the stator iron.

Mechanical losses (also known as *rotational losses*) include brush friction and bearing friction and windage (air friction). (Windage is a no-load loss but is not an electrical core loss.) Mechanical losses are determined by measuring the power input at the rated speed with no load. Stray losses are caused by nonuniform current distribution in the conductors. *Stray losses* are approximately 1% for DC machines and zero for AC machines.

Real power is only used to compute the *efficiency* of a rotating machine.

Equation 20: Efficiency

$$\eta = \frac{\text{output}}{\text{input}} = \frac{\text{output}}{\text{output} + \text{losses}} = \frac{\text{input} - \text{losses}}{\text{input}}$$

Regulation

The *voltage regulation*, VR, is

Equation 21: Voltage Regulation

$$\text{VR}=\frac{V_{nl} - V_{fl}}{V_{fl}} \times 100\%$$

The *speed regulation*, SR, is

Equation 22: Speed Regulation

$$\text{SR} = \frac{n_{nl} - n_{fl}}{n_{fl}} \times 100\%$$

No-Load Conditions

The meaning of the term no load is different for generators and motors. For unloaded shunt-wired alternators and generators, there is no electrical load connected across the output terminals, so although the field current flows, the line current, I, is zero. For unloaded shunt-wired motors, the work performed is zero, but line current is still drawn to keep the motor turning. All of the current is field current, however, and (neglecting mechanical losses) the armature current, I_a, is zero.

Equation 23: Generator No-Load Condition

$$I = 0;\ I_f \neq 0;\ I_a = I_f \quad \text{[generator]}$$

Equation 24: Motor No-Load Condition

$$I \neq 0;\ I_f = I;\ I_a = 0 \quad \text{[motor]}$$

Types of DC Machines

DC machines, both generators and motors, can be connected in a number of ways, as shown in Fig. 17. The arrows indicate the direction of the magnetic field. If the arrows have identical directions, the magnetic fields aid one another (the magnetomotive forces of the fields are additive). If the arrows are in opposite directions, the magnetic fields oppose one another (the magnetomotive forces of the fields are subtractive).

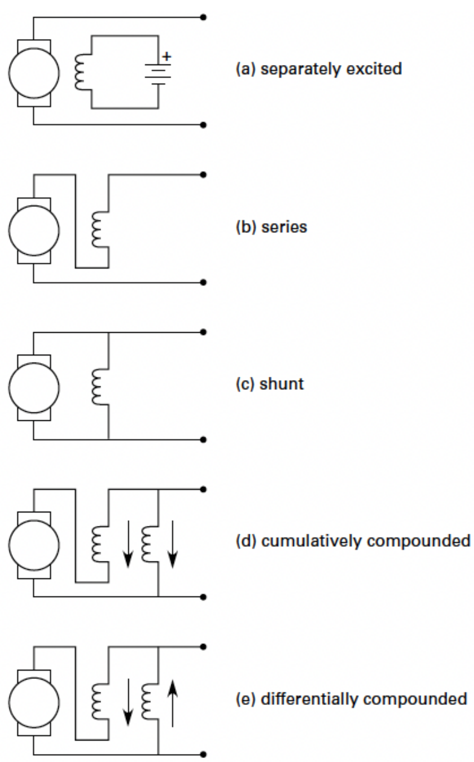
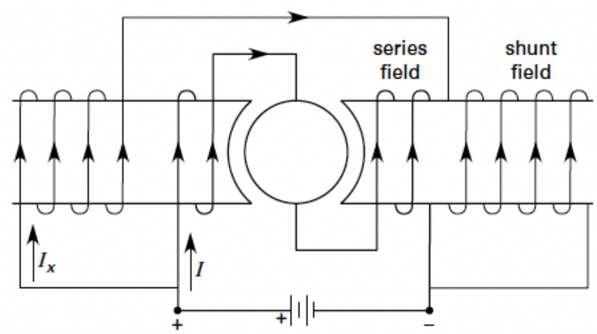


Figure 17: DC Machine Connections

The fields are shown as coils. Such coils have inductive and resistive components. Often variable resistors are added in series with the fields—especially the shunt field—to allow for control of the current through a given field. The current then determines the flux. The flux level controls either the generated voltage (in a generator) or the counter-electromotive force (in a motor), both of which impact the speed of the machine. In a generator, an external energy input maintains the speed constant. In a motor, the speed is generally allowed to change with the load.

Example 5

What type of machine and compounding are shown in the following illustration?



Solution

The machine has an external power source providing current to the windings, so this is a motor. A series and shunt winding are shown, which indicates the motor is either cumulatively compounded or differentially compounded. Use the direction of the arrows and the righthand rule to determine that the series and shunt field directions (magnetomotive forces) are additive. Therefore, this is a cumulatively compounded motor.

Series-Wired DC Machines

The equivalent circuit of a series-wired DC machine is shown in Fig. 18. The only components in the circuit are the field and armature resistances in series (from which the name is derived). The brush resistance is also considered to be in series but is often included in the armature resistance specification.

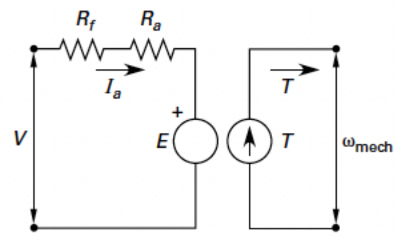


Figure 18: Series-Wired DC Motor Equivalent Circuit

The governing equations are Eq. 25 through Eq. 28. I_a is positive for a motor and negative for a generator. The magnetic flux varies with the armature current.

Equation 25: Series Equation

$$E = \kappa'_E \eta \phi = \kappa_E \eta I_a = V - I_a \left(R_a + R_f \right)$$

Equation 26: Series Equation

$$V = E + I_a \left(R_a + R_f \right)$$

The speed, torque, and current are related by the following.[18]

Equation 27: Series Torque, Current / Relationship

$$\frac{\mathrm{T}_1}{\mathrm{T}_2} = \left(\frac{I_{a,1}}{I_{a,2}} \right)^2 \approx \frac{n_1}{n_2}$$

The torque is as follows.

Equation 28: Series, Torque

$$T = \kappa'_T \Phi I_a^2 = \kappa_T \left(\frac{V}{\kappa_E \eta + R_A + R_f} \right)$$

For a motor, a reduction in load (torque) causes a corresponding reduction in armature current. However, because the field and armature currents are identical, the flux is reduced and the speed increases to maintain Eq. 26. Therefore, a series motor is not a constant speed device. A load

[18] The current, I, is directly related to the rotational speed, n. The torque, T, is related to the rotational speed squared, n^2, while the power, P, is related to the rotational speed cubed, n^3.

should never be completely removed from a running DC motor, and gears (not belts, which can slip) are the preferred method of connecting DC motors to their loads.

The back emf, E, is zero when the motor starts from rest. Therefore, the armature current, I_a, must be excessively high in order to keep Eq. 26 valid. For this reason, reduced voltages are required when starting, and the field resistance, R_f, is often a rheostat or switchable resistor bank.

At high speeds, the back emf, E, counteracts the applied voltage, increasing as the rotational speed increases. The stall speed is the speed at which Eq. 29 is valid.

$$E = I_a\left(R_a + R_f\right) = \frac{V}{2} \quad \text{[stall]}$$

Equation 29: Series, Stall Condition

Shunt-Wired DC Machines

Shunt-wired DC machines have a constant (but adjustable) magnetic field because the field current is constant. For shunt-wired motors, this results in a relatively constant speed. The magnetic coil is fed from the same line as the armature (as it is in Fig. 19) in a *self-excited machine*; in a *separately excited machine* the field coil is fed from another source. In Eq. 30 through Eq. 36, I_a is positive for a motor and negative for a generator.

Equation 30: Shunt Equation

$$E = \kappa_E \eta \Phi$$

Equation 31: Shunt Equation

$$V = E + I_a R_a = I_f R_f$$

Equation 32: Shunt Motor Equation

$$I = I_a + I_f \quad \text{[motor]}$$

Equation 33: Shunt Generator Equation

$$I = I_a - I_f \quad \text{[generator]}$$

The speed-current relationship follows.

Equation 34: Shunt Speed-Current

$$n = n_{nl} - \kappa_n T = \frac{V - I_a R_a}{\kappa_E \Phi}$$

The torque is as follows.

Equation 35: Shunt Torque

$$T = \kappa_T \Phi I_a$$

The torque and current are proportional.

Equation 36: Shunt Torque, Current / Relationship

$$\frac{T_1}{T_2} = \frac{I_{a,1}}{I_{a,2}}$$

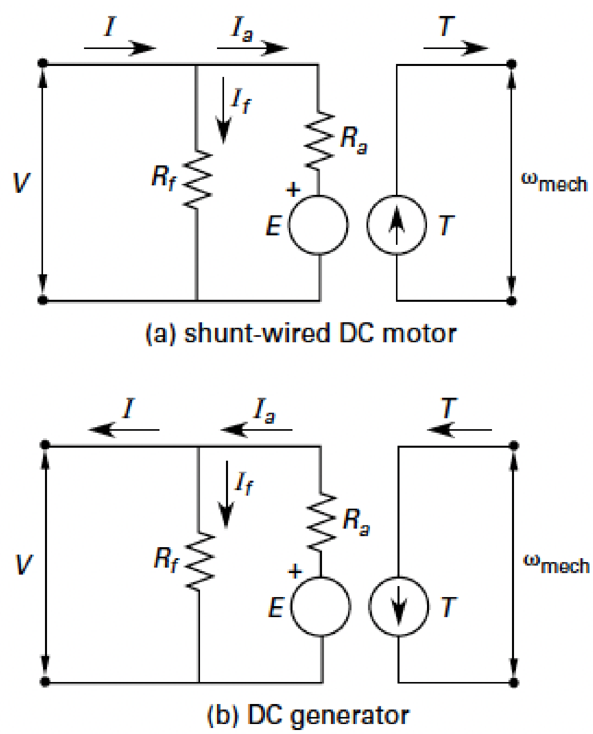


Figure 19: Shunt Equivalent Circuits

Compound DC Machines

Compound DC machines have both series and shunt windings. Their performance is between those of shunt and series machines.

Example 6

A two-pole DC generator with a lap-wound armature is turned at 1800 rpm. There are 100 conductors between the brushes. The average magnetic flux density in the air gap between the pole faces and armature is 1.2 T. The pole faces have an area of 0.03 m^2.

What is the no-load terminal voltage?

Solution

The flux per pole is

$$\Phi = BA = (1.2\ \mathrm{T})(0.03\ \mathrm{m}^2) = 0.036\ \mathrm{Wb}$$

The term "no load" for a generator means the line current is zero. Also, $N = z/a = z/p$, for a lap-wound armature, that is $a = p$.

$$E = \frac{zp\Phi n}{60a} = \frac{(100)(2)(0.036\ \mathrm{Wb})(1800\ \mathrm{rev/min})}{\left(60\ \frac{\mathrm{s}}{\mathrm{min}}\right)(2)} = 108\ \mathrm{V}$$

Voltage-Current Characteristics for DC Generators

Figure 20 illustrates the voltage-current characteristics for a DC Generator.

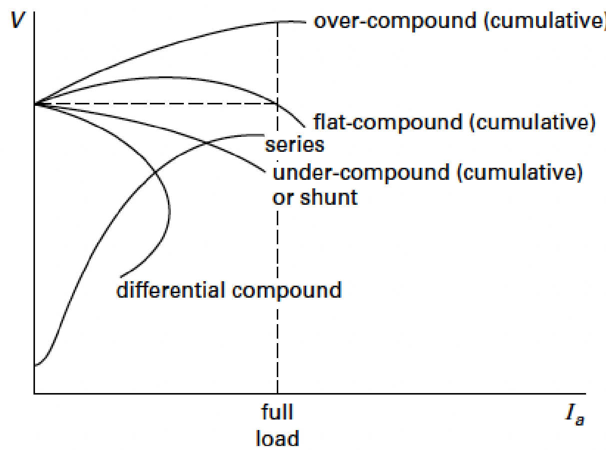


Figure 20: DC Generator Voltage-Current Characteristics

Torque Characteristics for DC Motors

The torque produced by a DC Motor is illustrated in Fig. 21.

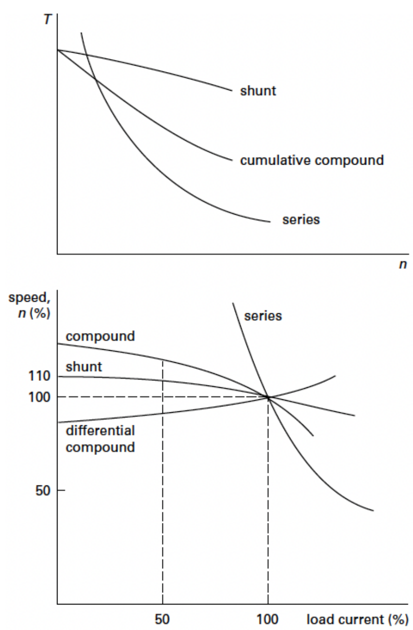


Figure 21: DC Motor Torque Characteristics

Starting DC Motors

DC motors have very low armature resistance. At rest, there is no back emf, *E* [CEMF]. If connected across the full line voltage, the high current could damage the motor. Such motors are almost always started with a resistance in series with the armature winding.[19] An initial resistance is chosen that limits the starting current to approximately 150% of the full-load current. As the motor builds up speed, the back emf opposes the line voltage, reducing the current. The starting resistance can then be gradually reduced and removed.

Because torque depends on the current, starting torque is limited to approximately 225%, 175%, and 150% of the full-load torque for series, compound, and shunt motors, respectively, when the starting current is 150% of full-load current.

Speed Control for DC Motors

The speed of a DC motor can be controlled by changing the armature conditions, field conditions, or both. Changes can be made manually or automatically. *Armature control* techniques include (a) placing a variable resistance in series or parallel with the armature and (b) changing the voltage across the armature. In all control techniques, the field voltage is held constant. With a constant torque load, the DC motor speed varies in approximate proportion to armature voltage changes.

Two *field control* (*field weakening*) techniques include (a) changing the resistance of the field winding (series or shunt) and (b) changing the voltage across the field. In all field control techniques, the armature voltage is held constant. The DC motor speed increases when the flux is reduced, which happens when the field current is reduced.

To increase the rotational speed of a DC motor, the resistance of the field winding is increased. Increasing the resistance of the field winding causes a sequence of changes to the current, flux, generated emf, *and* torque, which changes the rotational speed of the DC motor, as shown in Fig. 22. Figure 22(a) shows the responses of variables within a DC motor, and Fig. 22(b) shows the sequence of changes of those variables. The magnitude of each change can be calculated using the applicable equations given earlier. Similar principles apply when changing the voltage across the field.

[19] Small fractional horsepower (1/4 hp or smaller) motors are an exception.

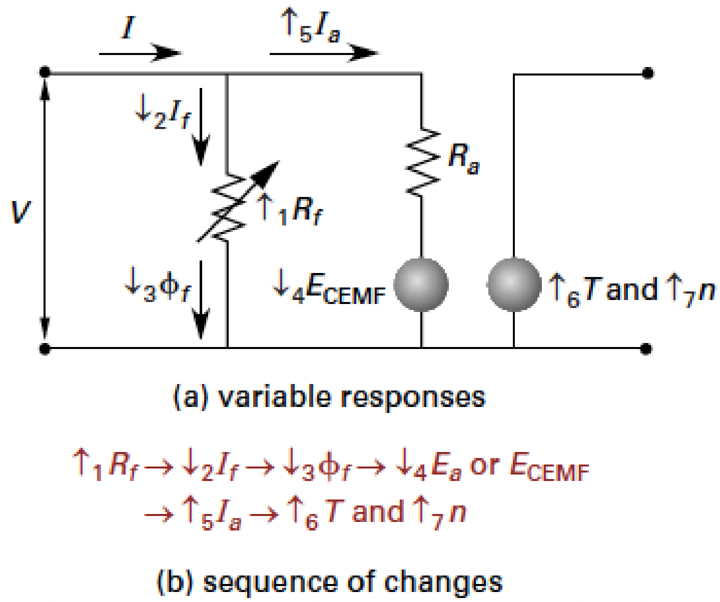


Figure 22: DC Motor Speed Control

Direction of Rotation of DC Motors

A DC motor's *direction of rotation can be reversed either by changing the direction of the armature's current OR by changing the direction of current in both the series and shunt fields*. If both the armature current and field current directions are changed, the DC motor will continue to rotate in the same direction as before the change.

REFERENCES

Items (latest editions) in **bold** are highly recommended for in-depth study.

A. **Camara, John A. *PE Power Reference Manual*. Belmont, CA: PPI (Kaplan), 2021.**

B. Earley, Mark, ed. *NFPA 70, National Electrical Code Handbook*. Quincy, Massachusetts: NFPA, 2020.

NOTE

Electrical refers to something related to electricity while "electric" refers to a device or machine that runs on electricity. Nevertheless, the NEC is sometimes referred to as the National Electric Code.

C. IEEE 315-1975. *Graphic Symbols for Electrical and Electronics Diagrams.* New York: IEEE, approved 1975, reaffirmed 1993.

D. IEEE 280-2021. *IEEE Standard Letter Symbols for Quantities Used in Electrical Science and Electrical Engineering.* New York: IEEE.

E. Grainger, John J., and William Stevenson, Jr. *Power System Analysis*. New York, McGraw Hill, 1994.

F. **Parker, Sybil P., editor in chief. *McGraw-Hill Dictionary of Scientific and Technical Terms,* 5th ed. New York, McGraw-Hill, 1994.**

G. **Wildi, Theodore. *Electrical Power Technology.* New York: John Wiley & Sons, 1981.**

Appendix A: Equivalent Units Of Derived And Common SI Units

Symbol	Equivalent Units			
A	C/s	W/V	V/Ω	$\mathrm{J/(s\cdot V)}$
C	$\mathrm{A\cdot s}$	J/V	$\mathrm{(N\cdot m)/V}$	$\mathrm{V\cdot F}$
F	C/V	$\mathrm{C^2/J}$	s/Ω	$\mathrm{(A\cdot s)/V}$
F/m	$\mathrm{C/(V\cdot m)}$	$\mathrm{C^2/(J\cdot m)}$	$\mathrm{C^2/(N\cdot m^2)}$	$\mathrm{s/(\Omega\cdot m)}$
H	W/A	$\mathrm{(V\cdot s)/A}$	$\mathrm{\Omega\cdot s}$	$\mathrm{(T\cdot m^2)/A}$
Hz	1/s	$\mathrm{s^{-1}}$	cycles/s	$\mathrm{radians/(2\pi\cdot s)}$
J	$\mathrm{N\cdot m}$	$\mathrm{V\cdot C}$	$\mathrm{W\cdot s}$	$\mathrm{(kg\cdot m^2)/s^2}$
$\mathrm{m^2/s^2}$	J/kg	$\mathrm{(N\cdot m)/kg}$	$\mathrm{(V\cdot C)/kg}$	$\mathrm{(C\cdot m^2)/(A\cdot s^3)}$
N	J/m	$\mathrm{(V\cdot C)/m}$	$\mathrm{(W\cdot C)/(A\cdot m)}$	$\mathrm{(kg\cdot m)/s^2}$
$\mathrm{N/A^2}$	$\mathrm{Wb/(N\cdot m^2)}$	$\mathrm{(V\cdot s)/(N\cdot m^2)}$	T/N	$\mathrm{1/(A\cdot m)}$
Pa	$\mathrm{N/m^2}$	$\mathrm{J/m^3}$	$\mathrm{(W\cdot s)/m^3}$	$\mathrm{kg/(m\cdot s^2)}$
Ω	V/A	$\mathrm{W/A^2}$	$\mathrm{V^2/W}$	$\mathrm{(kg\cdot m^2)/(A^2\cdot s^3)}$
S	A/V	1/Ω	$\mathrm{A^2/W}$	$\mathrm{(A^2\cdot s^3)/(kg\cdot m^2)}$
T	$\mathrm{Wb/m^2}$	$\mathrm{N/(A\cdot m)}$	$\mathrm{(N\cdot s)/(C\cdot m)}$	$\mathrm{kg/(A\cdot s^2)}$
V	J/C	W/A	C/F	$\mathrm{(kg\cdot m^2)/(A\cdot s^3)}$
V/m	N/C	$\mathrm{W/(A\cdot m)}$	$\mathrm{J/(A\cdot m\cdot s)}$	$\mathrm{(kg\cdot m)/(A\cdot s^3)}$
W	J/s	$\mathrm{V\cdot A}$	V^2/Ω	$\mathrm{(kg\cdot m^2)/s^3}$
Wb	$\mathrm{V\cdot s}$	$\mathrm{H\cdot A}$	$\mathrm{T/m^2}$	$\mathrm{(kg\cdot m^2)/(A\cdot s^2)}$

Appendix B: Physical Constants

Table Note 1

Quantity	Symbol	US Customary	SI Units
Charge			
electron	e		-1.6022×10^{-19} C
proton	p		$+1.6022 \times 10^{-19}$ C
Density			
air [STP][32°F, (0°C)]		0.0805 lbm/ft^3	1.29 kg/m^3
air [70°F, (20°C), 1 atm]		0.0749 lbm/ft^3	1.20 kg/m^3
sea water		64 lbm/ft^3	1025 kg/m^3
water [mean]		62.4 lbm/ft^3	1000 kg/m^3
Distance			
Earth radius[2]	⊕	2.09×10^{7} ft	6.370×10^{6} m
Earth-Moon separation[2]	⊕☾	1.26×10^{9} ft	3.84×10^{8} m
Earth-Sun separtion[2]	⊕⊙	4.89×10^{11} ft	1.49×10^{11} m
Moon radius[2]	☾	5.71×10^{6} ft	1.74×10^{6} m
Sun radius[2]	⊙	2.28×10^{9} ft	6.96×10^{8} m
first Bohr radius	a_0	1.736×10^{-10} ft	5.292×10^{-11} m
Gravitational Acceleration			
Earth [mean]	g	32.174 (32.2) ft/sec^2	9.8067 (9.81) m/s^2
Mass			
atomic mass unit	μ or m_μ $\frac{1}{12}m\left(^{12}C\right)$	3.66×10^{-27} lbm	1.6606×10^{-27} kg or 10^{-3} kg mol^{-1} / N_A or 931.481 MeV
Earth[2]	⊕	4.11×10^{23} slugs	6.00×10^{24} kg
Earth [customary U.S.][2]	⊕	1.32×10^{25} lbm	-
Moon[2]	☾	1.623×10^{23} lbm	7.36×10^{22} kg
Sun[2]	⊙	4.387×10^{30} lbm	1.99×10^{30} kg
electron rest mass	m_e	2.008×10^{-30} lbm	9.109×10^{-31} kg [0.511 MeV]
neutron rest mass	m_n	3.693×10^{-27} lbm	1.675×10^{-27} kg [939.6 MeV]
proton rest mass	m_p	3.688×10^{-27} lbm	1.672×10^{-27} kg [938.2 MeV]

Quantity	Symbol	US Customary	SI Units
Pressure			
atmospheric		14.696 (14.7) lbf/in^2	1.0133□10^5 Pa
Temperature			
standard		32°F (492°R)	0°C (273 K)
absolute zero		–459.67°F (0°R)	–273.16°C (0 K)
Velocity[3]			
Earth escape		3.67□10^4 ft/sec	1.12□10^4 m/s
light (vacuum)	c, c_0	9.84□10^8 ft/sec	2.9979 (3.00) □10^8 m/s
sound [air, STP]	a	1090 ft/sec	331 m/s
sound [air, 70°F, (20°C), 1 atm]		1130 ft/sec	344 ft/s
Volume			
Volume: molal ideal gas (STP)[4]		359 ft^3 / lbmol	22.41 m^3/kmol

Table 1 Notes

1. Units come from a variety of sources, but primarily from the Handbook of Chemistry and Physics, The Standard Handbook for Aeronautical and Astronautical Engineers, and the Electrical Engineering Reference Manual for the PE Exam. See also the NIST website at https://pml.nist.gov/cuu/Constants/.
2. Symbols shown for the solar system are those used by NASA. See https://science.nasa.gov/resource/solar-system-symbols/.
3. Velocity technically is a vector. It has direction.
4. The unit "lbmol" is an actual unit, not a misspelling.

Appendix C: Fundamental Constants

Quantity	Symbols	US Customary	SI Units
Avogadro's number	N_A, L		6.022×10^{23} mol^{-1}
Bohr magneton	μ_B		9.2732×10^{-24} J/T
Boltzmann constant	κ	5.65×10^{-24} ft-lbf/ R	1.3805×10^{-23} J/T
electron volt: $\left(\frac{e}{C}\right)$ J	eV		1.602×10^{-19} J
Faraday constant, $N_A e$	F		96485 C/mol
fine structure constant, inverse α^{-1}	α α^{-1}		7.297×10^{-3} (≈1/137) 137.035
gravitational constant	g_c	32.174 lbm-ft/lbf-sec^2	
Newtonian gravitational constant	G	3.44×10^{-8} ft^4 / lbf-sec^4	6.672×10^{-11} N·m^2 / kg^2
nuclear magneton	μ_N		5.050×10^{-27} J/T
permeability of a vacuum	μ_0		1.2566×10^{-6} N/A^2 (H/m)
permittivity of a vacuum, electric constant $1/\mu_0 c^2$	ε_0		8.854×10^{-12} C^2 / N·m^2 (F/m)
Planck's constant	h		6.6256×10^{-34} J·s
Planck's constant: h/2π			1.0546×10^{-34} J·s
Rydberg constant	R_∞		1.097×10^{7} m^{-1}
specific gas constant, air	R	53.3 ft-lbf/lbm- R	287 J/kg·K
Stefan-Boltzmann constant		1.71×10^{-9} BTU/ft^2-hr-°R^4	5.670×10^{-8} W/m^2·K^4
triple point, water		32.02 F, 0.0888 psia	0.01109 C, 0.6123 kPa
universal gas constant	R*	1545 ft-lbf/lbmol- R 1.986 BTU/lbmol- R	8314 J/kmol·K

Table Notes

1. Units come from a variety of sources, but primarily from the Handbook of Chemistry and Physics, The Standard Handbook for Aeronautical and Astronautical Engineers, and the Electrical Engineering Reference Manual for the PE Exam. See also the NIST website at https://pml.nist.gov/cuu/Constants/. The unit in Volume of "lbmol" is an actual unit, not a misspelling.

Appendix D: Mathematical Constants, Signs/Symbols, Maxwell's Equations

Quantity	Symbol	Value
Archimedes' constant (pi)	π	3.1415926536
base of natural logs	e	2.7182818285
Euler's constant	C or τ	0.5772156649

Signs/Symbols	Meaning
$\cdot$	multiplied by
/	divided by
:	ratio
$\gg$	much greater than
$\ll$	much less than
=	equals
$\equiv$	identical with
$\sim$	similar to
$\approx$	approximately equals
$\cong$	approximately equals, congruent
$\rightarrow$, $\doteq$	approaches
$\propto$	proportional, varies as
$\therefore$	therefore

Maxwell's Equations

integral form	point form	remarks
$\oint_s \mathbf{D}\cdot ds = \int_V \rho\, dv$	$\nabla\cdot\mathbf{D} = \rho$	Gauss' law
$\oint_s \mathbf{B}\cdot ds = 0$	$\nabla\cdot\mathbf{B} = 0$	nonexistence of magnetic monopoles
$\oint \mathbf{E}\cdot dl = \int_s \left(\frac{-\partial\mathbf{B}}{\partial t}\right)\cdot ds$	$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$	Faraday's law
$\oint \mathbf{H}\cdot dl = \int_s \left(\mathbf{J}_c + \frac{\partial\mathbf{D}}{\partial t}\right)\cdot ds$	$\nabla\times\mathbf{H} = \mathbf{J}_c + \frac{\partial\mathbf{D}}{dt}$	Ampère's law

Free-Space Form

integral form	point form
$\oint_s \mathbf{D}\cdot ds = 0$	$\nabla\cdot\mathbf{D} = 0$
$\oint_s \mathbf{B}\cdot ds = 0$	$\nabla\cdot\mathbf{B} = 0$
$\oint \mathbf{E}\cdot dl = \int_s \left(\frac{-\partial\mathbf{B}}{\partial t}\right)\cdot ds$	$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$
$\oint \mathbf{H}\cdot dl = \int_s \left(\frac{\partial\mathbf{D}}{\partial t}\right)\cdot ds$	$\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{dt}$

Electromagnetic Field Vector Equations

$$\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0(1+\chi_e)\mathbf{E}$$
$$\mathbf{B} = \mu\mathbf{H} = \mu_0\mathbf{H} + \mu_0\mathbf{M} = \mu_0(1+\chi_m)\mathbf{H}$$
$$\mathbf{J} = \sigma\mathbf{E} = \rho\mathrm{v}$$

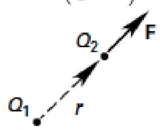
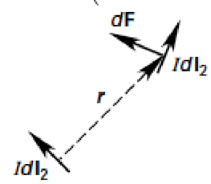
Appendix E: The Greek Alphabet

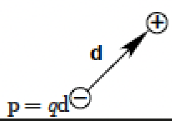
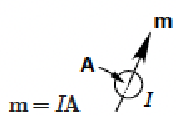
Α	α	alpha	Ν	υ	nu
Β	β	beta	Ξ	ξ	xi
Γ	γ	gamma	Ο	o	omicron
Δ	δ	delta	Π	π	pi
Ε	ε	epsilon	Ρ	ρ	rho
Ζ	ζ	zeta	Σ	σ	sigma
Η	η	eta	Τ	τ	tau
Θ	θ	theta	Υ	υ	upsilon
Ι	ι	iota	Φ	ϕ	phi
Κ	κ	kappa	Χ	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
Μ	μ	mu	Ω	ω	omega

Appendix F: SI Prefixes

symbol	prefix	value
a	atto	10^{-18}
f	femto	10^{-15}
p	pico	10^{-12}
n	nano	10^{-9}
μ	micro	10^{-6}
m	milli	10^{-3}
c	centi	10^{-2}
d	deci	10^{-1}
da	deka	10
h	hecto	10^{2}
k	kilo	10^{3}
M	mega	10^{6}
G	giga	10^{9}
T	tera	10^{12}
P	peta	10^{15}
E	exa	10^{18}

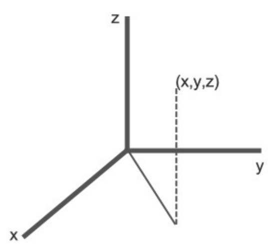
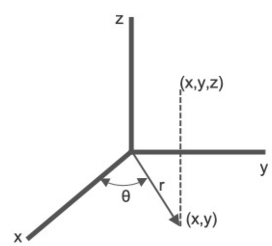
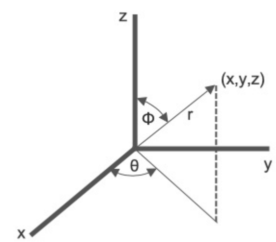
Appendix G: Comparison of Electric & Magnetic Equations

equation description	electric version	magnetic version	remarks
experimental force law	Coulomb's law $\mathbf{F} = \left(\frac{Q_1 Q_2}{4\pi\epsilon r^2}\right)\mathbf{r}$ 	force between two current elements $d\mathbf{F} = \left(\frac{\mu_0}{4\pi}\right)\left(\frac{I_2 d\mathbf{l}_2 \times (I_1 d\mathbf{l}_1 \times r)}{r^2}\right)$ 	The term Idl in the magnetic column is the equivalent of a "magnetic charge" q_m. The I or the dl can be the vector. The r is a unit vector pointing from 1 to 2.
field definitions from force law	$\mathbf{F} = Q\mathbf{E}$	$d\mathbf{F} = \mathbf{I}\times\mathbf{B}\,dl$ current element $d\mathbf{F} = \mathbf{J}\times\mathbf{B}\,dV$ distributed current element $d\mathbf{F} = q\,\mathbf{v}\times\mathbf{B}$ moving charge	The V used in this row represents volume, not voltage. The v is the velocity.
general force law	$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$ $d\mathbf{F} = (\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})\,dV$ where $dQ = \rho\,dV$		The V in this row represents the volume, not voltage. The v is the velocity.
definition of scalar and vector potential	$E = -\nabla V$	$\mathbf{B} = \nabla \times \mathbf{A}$	A is the magnetic vector potential.
Poisson's equation for the potential function	$\nabla^2 V = -\frac{\rho}{\epsilon}$	$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$	From a knowledge of the charge distribution, the potential can be found and then the E and B fields determined.
Gauss's law enclosing charge and Ampère's law enclosing current	$\iint \mathbf{D}\cdot d\,\mathbf{A} = \iiint \rho\,dV = Q$ $\nabla\cdot\mathbf{D} = \rho$	$\oint H\cdot d\mathbf{l} = I$ $\nabla\times\mathbf{H} = \mathbf{J}$	The V in this row represents volume.
constitutive relations	$\mathbf{D} = \epsilon\mathbf{E}$ $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$	$\mathbf{B} = \mu\mathbf{H}$ $\mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M}$	The second set of equations is always valid. The first set assumes the medium is linear and isotropic.
definitions of relative permittivity and permeability	$\epsilon_r = \frac{\epsilon}{\epsilon_0}$ $\epsilon_0 = 8.854\times10^{-12}$ F/m	$\mu_r = \frac{\mu}{\mu_0}$ $\mu_0 = 4\pi\times10^{-7}$ H/m	

equation description	electric version	magnetic version	remarks
capacitance and inductance of a field cell	$\epsilon_0 = \frac{C}{l}$	$\mu_0 = \frac{L}{l}$	Field cells are a construct designed to represent free space in terms of a parallel plate capacitor and an inductor. This capacitance and inductance exist regardless of the presence of an electric or magnetic field.
capacitance and inductance	$C = \frac{Q}{V}$	$L = \frac{\Lambda}{I}$	Λ is the flux linkage.
energy density of a field	$U = \frac{1}{2}\epsilon E^2$	$U = \frac{1}{2}\mu H^2$	Both energy and momentum are carried by a field.
energy stored by capacitance and inductance	$W = \frac{1}{2}CV^2$	$W = \frac{1}{2}LI^2$	
electromotive and magnetomotive force with sources present	$\oint \mathcal{E} \cdot d\mathbf{l} = \mathcal{E} = V$	$\oint \mathbf{H} \cdot d\mathbf{l} = NI = F_m = V_m$	The $\mathcal{E}$ is the emf, not the permittivity. Without sources present, both line integrals are equal to zero.
dipole moments			
dipole torque	$\mathbf{T} = \mathbf{p} \times \mathbf{E}$	$\mathbf{T} = \mathbf{m} \times \mathbf{B}$	This torque occurs due to the dipole being immersed in an external **E** or **B** field.
dipole potential energy	$W = -\mathbf{p} \cdot \mathbf{E}$	$W = -\mathbf{m} \cdot \mathbf{B}$	

electric	magnetic
emf $= V = IR$	mmf $= V_m = \phi \mathcal{R}$
current I	flux ϕ
emf $\mathcal{E}$ or V	mmf V_m
resistance $R = \rho l/A = l/\sigma A$	reluctance $\mathcal{R} = l/\mu A$
resistivity ρ	reluctivity $1/\mu$
conductance $G = 1/R$	permeance $P_m = \mu A/l$
conductivity $\sigma = 1/\rho$	permeability μ

Appendix H: Coordinate Systems and Related Operations

Mathematical Operations	Rectangular Coordinates	Cylindrical Coordinates	Spherical Coordinates
Conversion to Rectangular Coordinants	 $x = x$ $y = y$ $z = z$	 $x = r\cos\theta$ $y = r\sin\theta$ $z = z$	 $x = r\sin\phi\cos\theta$ $y = r\sin\phi\sin\theta$ $z = r\cos\phi$
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$	$\nabla f = \frac{\partial f}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{\theta} + \frac{\partial f}{\partial z}\mathbf{k}$	$\nabla f = \frac{\partial f}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial f}{\partial \phi}\boldsymbol{\phi} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \theta}\boldsymbol{\theta}$
Divergence	$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$	$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial (A_\phi \sin\phi)}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial A_\theta}{\partial \theta}$
Curl	$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{1}{r}\mathbf{r} & \boldsymbol{\theta} & \frac{1}{r}\mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & A_\theta & A_z \end{vmatrix}$	$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{1}{r^2\sin\theta}\mathbf{r} & \frac{1}{r^2\sin\theta}\boldsymbol{\phi} & \frac{1}{r}\boldsymbol{\theta} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ A_r & rA_\phi & rA\theta A_\phi \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\nabla^2 f = \frac{1}{r}\frac{\partial r}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}$	$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) + \frac{1}{r^2\sin^2\phi}\left(\frac{\partial^2 f}{\partial \theta^2}\right)$

