A SunCam Online Continuing Education Course

Motors/Generators/NEC

Course I—Rotating DC Machinery
Course II—Rotating AC Machinery
Course III—NEC: Motors & Generators

Notational Methods/Rotating AC Machines/NEMA Classifications/AC Potential Synchronous Machines/Induction Machines/Motor Types/Speed Control

by

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Nomenclature[1]

a	number of parallel armature paths	-
a	ratio of transformation	-
A	area	m^2
B	magnetic flux density	T
B	susceptance	S
B, $\boldsymbol{B}$	magnetic field	Wb/m (T·m)
E	generated emf	V
E	energy	J
E	generated voltage	V[2]
f	electrical frequency	Hz
F,$\boldsymbol{F}$	force	N
G	conductance	S
I,$\boldsymbol{I}$	constant or rms current	A
I_B	magnetization (quadrature current)	A
I_G	in-phase component of exciting current	A
k	constant	various[3]
l,$\boldsymbol{L}$	length	m
L	loss(es)	-
n	rotational speed	rev/min
N	number of turns/items	-
N	number of series armature paths	-
p	number of poles or poles/phase	-
P	power or power loss	W
pf	power factor	-
q	number of loops	-
R	resistance	Ω
r	radius	m
r, R	resistance	Ω
s	slip	-
S	apparent power	VA

[1] Not all the nomenclature, symbols, or subscripts may be used in this course—but they are related and may be found when reviewing the references listed for further information. Further, all the nomenclature, symbols, or subscripts will be found in of many electrical courses (on SunCam, PDH Academy, and also in many texts). For guidance on nomenclature, symbols, and electrical graphics: IEEE 280-2021, IEEE Standard Letter Symbols for Quantities Used in Electrical Science and Electrical Engineering, New York: IEEE; and IEEE 315-1975, Graphic Symbols for Electrical and Electronics Diagrams, New York: IEEE, approved 1975, reaffirmed 1993.

[2] Generated voltages are traditionally represented as "E" as the symbol but often "V" is used in everyday contexts. The international standard symbol is "U". The usage varies with the text.

[3] Rotating machines have numerous constants with differing names and subscripts. Anytime a subscript or superscript changes on the symbology for the constant, some different term (or possibly units) has come into play. Constants include torque (or motor), back-emf (or electrical or voltage), electrical time, mechanical time, power. See Table 1.

SCR	short-circuit ratio	-
SR	speed regulation	-
S_R	slew rate	V/s
SR	speed regulation	%
t	time	s
T	torque	N□m (ft-lbf)
T	period	s
v	variable voltage	V
V	constant or rms voltage	V
V	line voltage	V
v,**v**	velocity	m/s
V_0	generated voltage	V
VR	voltage regulation	%
X	inductance or reactance	Ω
Y	admittance	S
z	number of conductors	-
Z	impedance	Ω

Greek Symbols

δ	torque angle	rad
η	efficiency	-
θ	angle or phase angle difference	rad
κ	torque conversion factor	-
ϕ	angle	rad
ϕ	flux	Wb
Φ	magnetic flux	Wb
ω	angular frequency	rad/s
ω_{mech}	rotational speed	rad/s

Subscripts

0	initial	-
0	stator	-
1	equivalent stator	-
2	equivalent rotor	-
a	armature	-
adj	adjusted	-
aux	auxiliary	-
ave	average	-
b	blocked rotor	-
c	capacitor	-
CEMF	counter-electromotive force	-

cp	commutating pole	-
Cu	copper	-
d	direct	-
ds	direct axis	-
E	emf	-
e	electrical, emf	-
eff	effective	-
f	field	-
f	final	-
*fl,*fl	full load	-
g	generator	-
h	hysteresis	-
L	line, line-to-neutral, or load	-
m	mechanical or motor	-
max	maximum	-
mech	mechanical	-
net	net field	-
*nl,*fl	no load	-
n	rotational speed/torque[4]	-
oc	open circuit	-
p	phase	-
pf	power factor	-
pu	per unit	-
q	quadrature	-
r	rotor	-
rev/min	revolutions/minute	-
s	synchronous	-
sc	short circuit	-
st	stator	-
sync	synchronizing	-
t	terminal, total, torque	-
T	torque	-
x	armature	-

[4] Related to the inertia of the rotor. Sometimes shown as “*in*” for inertia. The constant converts torque to speed and is related to a motor’s resistance and voltage.

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COURSE REFERENCES

The theoretical information is primarily from one of the author's books, Ref. [A]. The NEC Ref. [B] is always a useful source for electrical engineers. Information useful in many aspects of electric engineering may be found in [C] and [D]. Reference [E] has detailed descriptions of analysis techniques. Reference [F] covers many terms in EE with excellent definitions and explanations. Reference [G] is one of the most comprehensive and best explained texts on motor and generator theory. The appendices (A-F) cover information useful in many engineering tasks with App. (G) providing a side by side comparison of electric and magnetic equations. Use these texts or their counterparts for indepth information. References in bold are highly recommended.

This course will focus on basics, that rarely change, and provide the basis for all other knowledge.

Rotating Machines

Rotating machines are broadly categorized as AC or DC machines. Both categories include machines that use power (i.e., motors) and those that generate power (alternators and generators). Most machines can be constructed in either single-phase or polyphase configurations, although single-phase machines may be outclassed in terms of economics and efficiency.

Types of small AC motors include split-phase, repulsion- induction, universal, capacitor, and series motors. Large AC motors are almost always three-phase, but it is necessary to analyze only one phase of the motor. Torque and power are divided evenly among the three phases. Machines can be wye- or delta-wired or both.

Wye connections have the following benefits.

• A neutral (ground) wire is intrinsically part of the circuit.
• Higher-order (harmonic) terms are not shorted out.
• Starting current is lower.

Nevertheless, high horsepower motors are usually run in delta. To avoid large starting currents, the motor can be started in wye and switched over to delta.

It is common to refer to line-to-line voltage as the *terminal voltage*, V.

Equation 1: Phase Voltage

$$V_p = \begin{cases} V_L & \text{[delta wired]} \\ \dfrac{V_L}{\sqrt{3}} & \text{[wye-wired]} \end{cases}$$

Equation 2: Phase Torque

$$T_p = \frac{T_t}{3}$$

Equation 3: Phase Power

$$P_p = \frac{P_t}{3}$$

Equation 4: Phase Apparent Power

$$S_p = \frac{S_t}{3}$$

Torque and Power

Torque and *power* are operating parameters. It takes power to turn an alternator or generator. A motor converts electrical power into mechanical power. In the SI system, power is given in kilowatts (kW). One horsepower is equivalent to 745.7 watts. The relationship between torque and power is

Equation 5: Torque-US Customary

$$T_{\text{ft-lbf}} = \frac{5252 P_{\text{horsepower}}}{\eta_{\text{rpm}}}$$

Equation 6: Torque-Metric

$$T_{\text{N}\cdot m} = \frac{1000 P_{\text{kW}}}{\omega_{\text{mech}}}$$

Equation 7: Induction Machines-Power, Torque, Speed

$$P_{\text{kW}} = \frac{T_{\text{N}\cdot\text{m}} \eta_{\text{rpm}}}{9549}$$

There are many important torque parameters for motors. The *starting torque* (also known as *static torque*, *breakaway torque*, and *locked-rotor torque*) is the turning effort exerted in starting a load from rest. *Pull-up torque* (*acceleration torque*) is the minimum torque developed during the period of acceleration from rest to full speed. *Pull-in torque* (as developed in synchronous motors) is the maximum torque that brings the motor back to synchronous speed. (Nominal pull-in torque is the torque that is developed at 95% of synchronous speed.) The *steady-state torque* must be provided to the load on a continuous basis. It establishes the temperature increase that the motor can withstand without deterioration. The *rated torque* is developed at rated speed and rated horsepower. The maximum torque a motor can develop at its synchronous speed is the *pull-out torque*. *Breakdown torque is the maximum torque the motor can develop without stalling (i.e., without coming rapidly to a complete stop*).

Equation 33.8 is the general torque expression for a rotating machine with N coils of cross-sectional area A, each carrying current I through a magnetic field of strength B.

Equation 8: Torque-Rotating Machine

$$T = NBAI \cos \omega t$$

Service Factor

The horsepower and torque ratings listed on the nameplate of a motor can be provided on a continuous basis without overheating. Motors can be operated at slightly higher loads without exceeding a safe temperature rise, but the higher temperature has a deteriorating effect on the winding insulation. (A general rule of thumb is that a motor loses two or three hours of useful life for each hour run at the factored load.) The ratio of the safe to standard loads is the *service factor*, usually expressed as a decimal or percent above rating. Service factors vary from 1.15 to 1.4, with the lower values going to larger, more efficient motors.

Equation 9: Service Factor

$$\text{service factor} = \frac{\text{safe load}}{\text{nameplate load}}$$

Motor Classifications

The National Electrical Manufacturers Association (NEMA) has categorized motors in several ways: *speed classification* (constant-, adjustable-, multi-, varying-speed, etc.); *service classification* (general, definite, and special purpose); and *motor class*. Motor class is a primary indicator of the maximum motor operating temperature, which, in turn, depends on the type of insulation used on the conductors. The motor classes are Class A, 105°C; Class B, 130°C; Class F, 155°C; and Class H, 180°C.

Power Losses

The losses for all rotating machines can be divided into four categories. *Copper losses*, P_{Cu}, are real power losses to wire and winding resistance. In a DC machine, copper losses occur in the armature and field windings as well as from the brush contact resistance. In an AC machine, copper losses occur in the armature and exciter field windings. There are no brush losses in an induction machine.

Equation 10: Copper Losses

$$P_{\text{Cu}} = \square\, I^2 R$$

In a motor, the rotor copper losses, $P_{r,\text{Cu}}$ mare related to the mechanical power output, P_m, and the slip, s, as shown in Eq. 11.

Equation 11: Induction Machine Rotor Copper Losses

$$P_m = (1 - s) P_{r,\text{Cu}}$$

Core losses, including hysteresis and eddy current losses, are constant losses that are independent of the load and, for that reason, are also known as *open-circuit* and *no-load losses*. In DC and synchronous AC machines, core losses occur in the armature iron. In induction machines, core losses occur in the stator iron.

Mechanical losses (also known as *rotational losses*) include brush and bearing friction and *windage* (air friction). (Windage is a no-load loss but is not an electrical core loss.) Mechanical losses are determined by measuring the power input at rated speed and no load.

Stray losses are caused by nonuniform current distribution in the conductors. Stray losses are approximately 1% for DC machines and zero for AC machines.

Real power only is used to compute the *efficiency* of a rotating machine.

Equation 12: Efficiency in General Terms

$$\eta = \frac{\text{output}}{\text{input}} = \frac{\text{output}}{\text{output} + \text{losses}} = \frac{\text{input} - \text{losses}}{\text{input}}$$

The percentage efficiency equation can also be written as

Equation 13: Efficiency in Power Terms

$$\% \text{ efficiency} = \frac{P_{\text{out}}}{P_{\text{in}}} \times 100\%$$

Regulation

The *voltage regulation*, VR, is

Equation 14: Voltage Regulation

$$\text{VR} = \frac{V_{\text{nl}} - V_{\text{fl}}}{V_{\text{fl}}} \times 100\%$$

The *speed regulation*, SR, is

Equation 15: Speed Regulation

$$\text{SR} = \frac{n_{\text{nl}} - n_{\text{fl}}}{n_{\text{fl}}} \times 100\%$$

No-Load Conditions

The meaning of the term *no load* is different for generators and motors. For unloaded shunt-wired alternators and generators, there is no electrical load connected across the output terminals, so although the field current flows, the line current, I, is zero. For unloaded shunt-wired motors, the work performed is zero, but line current is still drawn to keep the motor turning. All of the current is field current, however, and (neglecting mechanical losses) the armature current, I_a, is zero.

Equation 16: No-Load Generator

$$I = 0 \quad I_f \neq 0 \quad I_a = I_f \quad \text{[generator]}$$

Equation 17: No-Load Motor

$$I \neq 0 \quad I_f = I \quad I_a = 0 \quad \text{[motor]}$$

Production of AC Potential

A potential of alternating polarity is produced by an *alternator* (*AC generator*). A permanent magnet or DC electromagnet produces a constant magnetic field. Figure 1 illustrates how several loops of wire can be combined into a rotating *induction coil* or *armature* to produce a continuously varying potential in a *dynamo* (*coil dynamo*).[5] Three-phase alternators have three sets of independent windings. A single-phase AC alternator has only one set of windings.

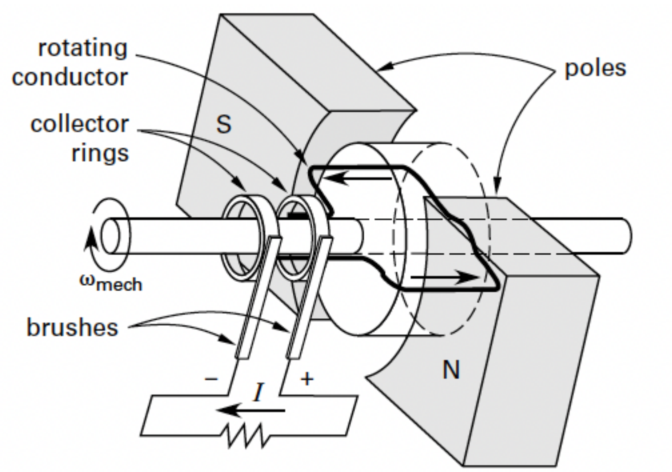


Figure 1: Elementary Two-Pole, Single-Coil Dynamo

The induced voltage, E, is commonly called *electromotive force* (emf). In an elementary alternator, emf is the desired end result and is picked off by stationary brushes making contact with slip rings on the rotating shaft.[6] In a motor, emf is also produced but is referred to as back emf (counter emf) because it opposes the input current.

Assuming the magnetic field flux density under each pole face, B, is uniform, the maximum flux linked by a coil with N turns and area A is NAB. Because the coil rotates, the flux linkage is a function of the projected coil area. The instantaneous induced voltage is predicted by Faraday's law. Care must be taken to distinguish between the armature speed, n (in rpm), the angular armature speed, ω_{mech} (in rad/s), and the linear and angular voltage frequencies, f and ω (in Hz and rad/s, respectively). Note the distinction in Eq. 18 between the rotational speeds of the armature ($\omega_{mech} = 2\pi n/60$) and the electrical waveform ($\omega = 2\pi f$).

[5] Dynamo refers to the change of mechanical energy to electric energy; although, dynamos originally were defined as those producing DC only—that is, using a commutator.

[6] Practical alternator use slip rings to feed the field, not to transfer the generated voltage.

Equation 18: AC Terminal Voltage

$$
\begin{aligned}
V(t) &= V_{\max}\sin\omega t = \omega NAB\sin\omega t \\
&= \left(\frac{p}{2}\right)\omega_{mech}NAB\sin\left(\left(\frac{p\omega_{\text{mech}}}{2}\right)t\right) \\
&= \frac{\pi npNAB}{60}\sin\left(\left(\frac{p\omega_{\text{mech}}}{2}\right)t\right)
\end{aligned}
$$

Equation 19: Effective Terminal Voltage

$$V = \frac{V_{\max}}{\sqrt{2}} = \frac{\omega NAB}{\sqrt{2}} = \frac{p\omega_{\text{mech}}NAB}{2\sqrt{2}} = \frac{\pi npNAB}{60\sqrt{2}} \quad \text{[effective]}$$

Table 1 summarizes the most frequently used formulas for a single-phase AC alternator.

Table 1: Single-Phase AC Alternator Formulas

	in terms of n	in terms of ω_{mech}	in terms of f	in terms of ω
n (rpm)	–	$\frac{30\,\omega_{\text{mech}}}{\pi}$	$\frac{120f}{p}$	$\frac{60\omega}{\pi p}$
ω_{mech} (mechanical rad/s)	$\frac{\pi n}{30}$	–	$\frac{4\pi f}{p}$	$\frac{2\omega}{p}$
f (Hz)	$\frac{pn}{120}$	$\frac{p\,\omega_{\text{mech}}}{4\pi}$	–	$\frac{\omega}{2\pi}$
ω (electrical rad/s)	$\frac{\pi pn}{60}$	$\frac{p\,\omega_{\text{mech}}}{2}$	$2\pi f$	–
single-phase $V_{\max}$ (V)	$\frac{\pi pnNAB}{60}$	$\frac{\omega_{\text{mech}}pNAB}{2}$	$2\pi fNAB$	ωNAB
single-phase V_{eff} (V)	$\frac{\pi pnNAB}{60\sqrt{2}}$	$\frac{\omega_{\text{mech}}pNAB}{2\sqrt{2}}$	$\frac{2\pi f\,NAB}{\sqrt{2}}$	$\frac{\omega NAB}{\sqrt{2}}$
single-phase V_{ave} (V)	$\frac{pnNAB}{30}$	$\frac{\omega_{\text{mech}}pNAB}{\pi}$	$4fNAB$	$\frac{2\omega NAB}{\pi}$

Alternators are characterized by the number of magnetic poles, p. (Both north and south poles are counted to distinguish the quantity from the number of *pole pairs*.) The coil pitch (pole pitch) is the angle between the poles, or 360° divided by *p*. A two-pole alternator produces one complete sinusoidal cycle per revolution. An alternator with *p* poles produces (1/2)*p* cycles per revolution. Because the armature normally turns at a constant speed known as the *synchronous speed*, n_s, the *electrical frequency*, *f*, of the generated potential is given by Eq. 21. The actual rotational speed, *n*, is known as the *mechanical frequency*.

Equation 20: Synchronous Speed, Induction Machines

$$n_s = \frac{120f}{p} = \frac{60\omega_{\text{mech}}}{2\pi} = \frac{60\omega}{\pi p} \quad \text{[synchronous speed]}$$

Equation 21: Frequency

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{pn_s}{120}$$

Because the coils in an alternator have inductance as well as resistance, the rated capacity of an AC machine is stated as apparent power at some rated voltage and power factor.

Example 1

A four-pole alternator produces a 60 Hz potential. What is the (a) mechanical speed of the armature, (b) angular velocity of the potential, and (c) angular velocity of the armature?

Solution

(a) From Eq. 20, the rotational speed is

$$\begin{aligned} n &= n_s \\ &= \frac{120f}{p} \\ &= \frac{\left(120\ \frac{\text{pole}\cdot\text{s}}{\text{min}}\right)(60\ \text{Hz})}{4\ \text{poles}} \\ &= 1800\ \text{rpm} \end{aligned}$$

(b) The angular velocity of the 60 Hz potential is

$$\begin{aligned} \omega &= 2\pi f \\ &= \left(2\pi\ \frac{\text{rad}}{\text{cycle}}\right)(60\ \text{Hz}) \\ &= 377\ \text{rad/s} \end{aligned}$$

(c) From Eq. 18, the angular velocity of the armature is

$$\begin{aligned}\omega_{\text{mech}} &= \frac{2\omega}{p} = \frac{2\pi n}{60} \\ &= \frac{\left(2\pi \ \frac{\text{rad}}{\text{rev}}\right)\left(1800 \ \frac{\text{rev}}{\text{min}}\right)}{60 \ \frac{\text{s}}{\text{min}}} \\ &= 188.5 \ \text{rad/s}\end{aligned}$$

Example 2

The rotor of a single-phase, four-pole alternator rotates at 1200 rpm. The effective diameter and length of the 20-turn (loop) coil are 0.12 m and 0.24 m, respectively. The magnetic flux density is 1.2 T. What is the effective voltage produced?

Solution

The coil area is

$$\begin{aligned}A &= 2rl \\ &= (0.12 \ \text{m})(0.24 \ \text{m}) \\ &= 0.0288 \ \text{m}^2\end{aligned}$$

From Eq. 19, the effective voltage is

$$\begin{aligned}V &= \frac{V_{\text{max}}}{\sqrt{2}} = \frac{\omega NAB}{\sqrt{2}} = \frac{p\omega_{\text{mech}}NAB}{2\sqrt{2}} = \frac{\pi npNAB}{60\sqrt{2}} \quad \text{[effective]} \\ &= \frac{p\omega_{\text{mech}}NAB}{2\sqrt{2}} \\ &= \frac{(4 \ \text{poles})\left(1200 \ \frac{\text{rev}}{\text{min}}\right)\left(2\pi \ \frac{\text{rad}}{\text{rev}}\right)\left(\frac{1 \ \text{min}}{60 \ \text{s}}\right)(20 \ \text{loops})\left(0.0288 \ \text{m}^2\right)(1.2 \ \text{T})}{2\sqrt{2}} \\ &= 122.8 \ \text{V}\end{aligned}$$

Armature Windings

An *armature winding* consists of several coils of continuous wire formed into q loops. Each loop in an armature contributes two *conductors*, also known as *bars*. The number of conductors, z, is

Equation 22: Conductors & Loops

$$z = 2q$$

Voltage is induced only in conductors and inductors that are parallel to the armature shaft (i.e., that cut magnetic flux lines). The number of *series paths*, N, between positive and negative brush sets depends on how the coils are wound and connected to the commutator.

Equation 23: Series Paths

$$N = \frac{z}{a} = \frac{2q}{a}$$

The number of parallel armature paths between each pair of brushes, a, equals the number of poles, p, in a *lap-wound armature* (*simplex-lap armature* or *multiple-drum armature*).[7] (The number of poles is also equal to the number of brushes.) The coil connections are made to adjacent commutator segments. This type of winding is commonly used in DC machines and induction motors and, to a lesser extent, in AC generators.

In a *wave-wound armature* (*two-circuit* or *series-drum armature*), $a = 2$, and the coil connections are on commutator segments on opposite sides of the armature regardless of the number of poles. Only two brushes are needed, although more can be used. This configuration is commonly used in DC armatures requiring the generation or use of higher voltages than lap windings can tolerate.

A *multiplex winding* has two or more distinct coils connected in parallel. The number of parallel current paths, a, in the armature can be determined by the type of winding and the number of poles, p, as indicated in Table 2.

Table 2: Multiplex Winding Current Paths

type	lap winding	wave winding
simplex	$a = p$	$a = 2$
duplex	$a = 2p$	$a = 4$
triplex	$a = 3p$	$a = 6$
quadriplex	$a = 4p$	$a = 8$

[7] Auxiliary windings are used in *shaded-pole motors*; *split-phase motors* have two separate windings; *capacitor motors* have capacitors in series with an auxiliary winding.

Practical Alternators

There are several reasons why large alternators are not designed with a stationary magnetic field and rotating coil as shown in Fig. 1.

- The rotating coils must be well insulated to prevent shorting with the high voltages that are induced.
- Structural bracing is required to counteract the large centrifugal force that results from rotating many coils of wire.
- It is difficult to make efficient high-voltage, high-power connections through slip rings.

For these reasons, practical alternators (see Fig. 2) reverse the locations of the field and induction coils so that the low-voltage field revolves and the high-voltage field is induced in stationary coils. The magnetic field is produced by *field windings* whose *DC magnetization current* is supplied through brushes and slip rings. Such an armature containing field coils is known as a *rotor* [rotating part). The stationary induction coils are placed in slots a small distance from the rotor and are known as the *stator* (stationary part). The closer the stator and rotor are, the smaller the magnetization current required.

With *cylindrical rotors* (*wound rotors*), the field windings are embedded in an externally smooth-surfaced cylindrical rotor. This makes them suitable for highspeed (i.e., greater than 1800 rpm) operation because windage losses are reduced. The poles of *salient-pole rotors*, on the other hand, resemble exposed wound electromagnets. This is adequate for 1800 rpm and less.

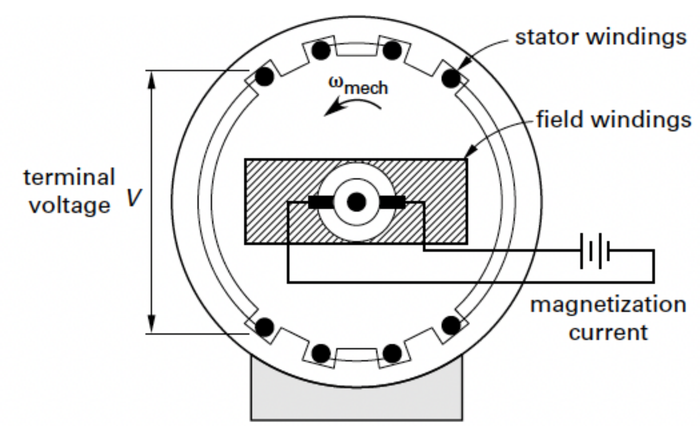


Figure 2: Practical Alternator

Rotating Magnetic Field

When a three-phase AC output is applied to a three-phase motor stator, the result is a magnetic field on the motor's stationary windings. The resultant magnetic field rotates with respect to those stationary windings. The speed of rotation is called the *synchronous speed*, n_s, in revolutions per minute (rpm) and is dependent on the number of poles per phase on the stator of the motor, p, and the applied frequency, f. This rotating magnetic field determines the speed of a synchronous motor and is responsible for the induced voltage, current, and torque in an induction motor rotor.

Equation 24: Synchronous Speed

$$n_s = \frac{120f}{p}$$

A three-phase current is shown in Fig. 3. The seven points beneath the currents represent one complete cycle. In Fig. 4, each of the points is applied to a two-pole machine. The magnetic field moves 60° between points, which correlates with the 60° between subsequent poles for a rotational speed of 3600 rpm.[8] As shown in Eq. 24 and Fig. 3, the correlation between the mechanical angles between poles does not need to be the same as the electrical angles of the three-phase source. In a four-pole machine, the electrical angle between the points of Fig. 3 would remain 60°, while the mechanical angle between points would be 30° with a subsequent rotational speed of 1800 rpm.

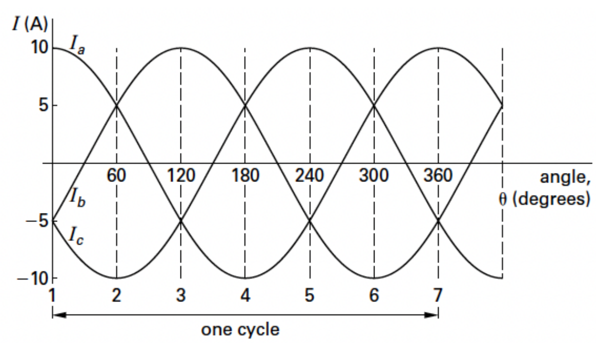


Figure 3: Three-Phase Currents

In Fig. 4, the direction of rotation for the magnetic field is clockwise. If any two of the three phases are reversed (the input leads are swapped), the rotation changes to counterclockwise. Therefore, to change the direction of an AC machine or its magnetic field, change the direction of the rotating field. For a three-phase machine, this is accomplished by swapping any two leads. For a single-

[8] The field moves 60° between points, or 1/6 of a revolution. In one cycle, one revolution occurs. A cycle is 1/60 of a second. Therefore, 60 revolutions occur in one second, for a synchronous speed of 3600 rpm.

phase machine, numerous methods exist; however, all methods involve changing the direction of the revolving magnetic field.

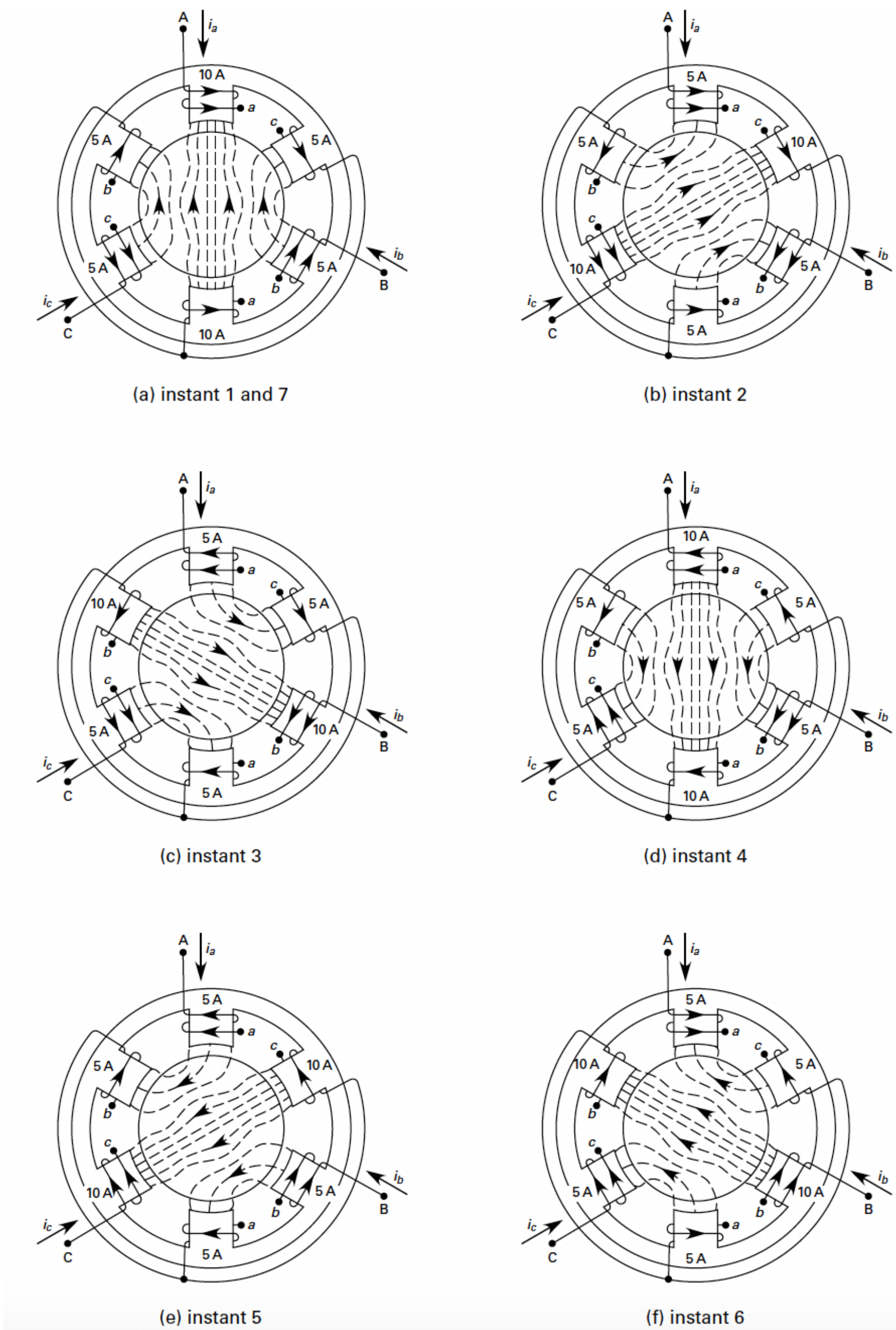


Figure 4: Rotating Magnetic Flux Patterns

Synchronous Motors

Synchronous motors are essentially dynamo alternators operating in reverse. Alternating current is supplied to the stationary stator windings. DC current is applied to the field windings in the rotor through brushes and slip rings as in an alternator.[9] The field current interacts with the stator field, causing the armature to turn. Because the stator field frequency is fixed, the motor runs only at a single synchronous speed (see Eq. 20).

Important features of synchronous motors follow.

- They turn at constant speeds, regardless of the load. (The speed may momentarily change when the load is changed.) Stalling occurs when a motor's counter-torque is exceeded.
- The power factor can be adjusted manually without losing synchronization by varying the field current. A unity power factor occurs with *normal excitation* current. The power factor is leading (lagging) when the current is more (less) than normal, and this is known as being *over*- (*under*-) *excited*.
- They can draw leading currents and be used for power factor correction, in which case they are known as *synchronous capacitors* (*synchronous condensers*).[10]

Because the starting torque is zero, it is necessary to bring a synchronous motor up to speed by some other means. The most common method is to include auxiliary windings in the pole faces so that the motor can be started as an induction motor. At synchronous speed, the auxiliary windings draw no current. If the motor speed becomes nonsynchronous, the auxiliary windings draw power to resynchronize the rotor.

Synchronous Machine Equivalent Circuit

Figure 5 illustrates a simple equivalent circuit for a synchronous machine. The vector voltage relationship is defined by Eq. 25 and Eq. 26, which introduce the equivalent synchronous inductance, X_s, of each phase. V_p is the phase voltage. The series armature resistance, R_a, is small and normally disregarded.

Equation 25: Synchronous Machine: Reactive Generator

$$\mathbf{E} = \mathbf{V}_p + \left(R_a + jX_s\right)\mathbf{I}_a \approx \mathbf{V}_p + jX_s\mathbf{I}_a \quad \text{[alternator]}$$

[9] The magnetization energy can also be generated through induction. In that case, the rotor includes diodes to rectify the induced AC potential.

[10] Overexcitation in submarine power plants using a motor-generator allows for power factor correction while still supplying the motor load. See Eq. 25 and Fig. 5.

Equation 26: Synchronous Machine: Motor

$$\mathbf{V}_p = \mathbf{E} + \left(R_a + jX_s\right)\mathbf{I}_a \approx \mathbf{E} + jX_s\mathbf{I}_a \quad \text{[motor]}$$

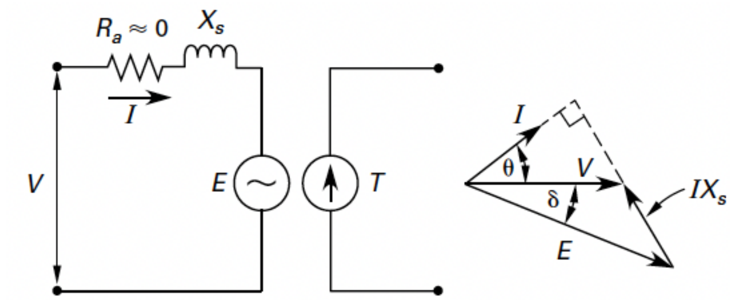


Figure 5: Synchronous Machine Equivalent Circuit

Equation 27 gives the real power generated per phase. For a motor, the power factor is determined by *E* and I_f. For an alternator, the power factor determines *E* and I_f. (Equation 27 mixes variables with different units.)

Equation 27: Real Power per Phase

$$P_p = T_p \omega_{\text{mech}} = VI\cos\theta = \left(\frac{VE}{X_s}\right)\sin\delta$$

Equation 28 gives the torque produced per phase. Φ_r and Φ_{st} are the internal rotor and stator fluxes, respectively. The *torque angle*, *δ* (also known as the *power angle* and *displacement angle*), is the phase angle difference between the applied voltage, *V*, and the generated emf, *E*. It is positive for an alternator and negative for a motor. Torque is maximum when δ = 90°, a condition equivalent to a unity power factor and known as *pull-out torque*. The *pull-out power* is found by setting *δ* = 90° in Eq. 27.

Equation 28: Synchronous Machine Torque

$$T_p = \kappa_T \Phi_r \Phi_{st} \sin\delta$$

The apparent power per phase is

Equation 29: Synchronous Machine Apparent Power

$$S = VI = \frac{P_p}{\cos\theta}$$

Example 3

A six-pole motor is connected to a three-phase, 240 V (rms), 60 Hz line. Its stator windings are connected in a wye configuration. The motor has a synchronous reactance of 3 Ω per phase and is rated at 10 kVA at its synchronous speed and 100% power factor.

What are the (a) synchronous speed, (b) phase voltage, (c) line current, (d) voltage drop across the synchronous reactance, (e) generated back emf, and (f) torque angle?

Solution

(a) Equation 24 gives the synchronous speed.

$$n_s = \frac{120f}{p} = \frac{(120)(60\text{ Hz})}{6} = 1200\text{ rev/min}$$

(b) Because the winding are in wye configuration, the phase voltage is

$$V_p = \frac{V}{\sqrt{3}} = \frac{240\text{ V}}{\sqrt{3}} = 138.6\text{ V}$$

(c) The line current is the same as the phase current, which can be calculated from the apparent power. Each phase draws one-third of the apparent power. The real power.

$$P_p = \left(\frac{S}{3}\right)\cos\theta = \frac{10{,}000\text{ VA}}{3} = 3333\text{ W}$$

$$I_p = \frac{P_p}{V_p} = \frac{3333\text{ W}}{138.6\text{ V}} = 24.05\text{ A}$$

(d) The voltage drop across each winding is

$$V_p = I_p X_p = (24.05\text{ A})(3\ \Omega) = 72.15\text{ V}$$

(e) The back emf is

$$\mathbf{E} = V_p - jIX = 138.6 - j72.15 = 156.3\ \text{V}\angle -27.50^\circ$$

(f) The torque angle was found in part (e) to be $\delta = -27.50^\circ$.

Synchronous Generator

A *synchronous generator equivalent circuit* is the reverse of the circuit shown in Fig. 6. That is, instead of the voltage generating a torque, the torque is used to spin the rotor's magnetic field and generate a current in the stator's stationary windings.

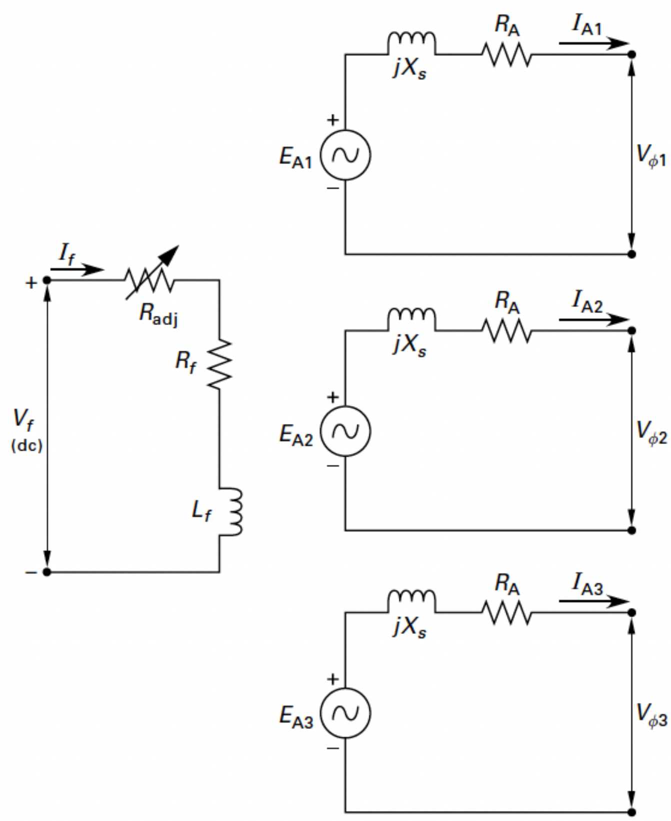


Figure 6: Synchronous Generator Equivalent Circuit

A single-phase model of a synchronous generator equivalent circuit is shown in Fig. 7.

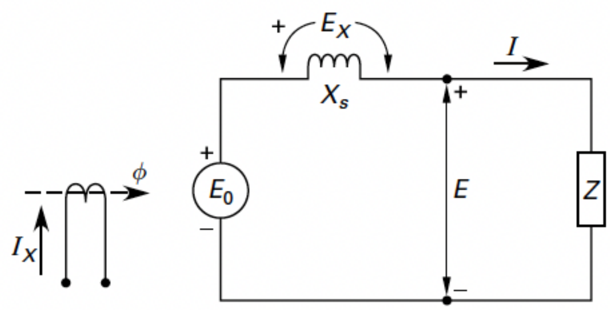


Figure 7: Single-Phase Synchronous Generator Equivalent Circuit Model

The term ϕ is the magnetic flux, which links the rotor field to the stator and generates the voltage, E_0. The term E_x is the armature reaction voltage, which reduces the voltage output, E, and/or misaligns the main field— as measured by the torque angle, δ, between E_0 and E. The impact of the armature reaction is measured by the synchronous reactance, X_s. The output voltage, E, is the single-phase line-to-neutral voltage.

A synchronous generator output is driven by two reactances, one direct (X_d) and the other in quadrature (X_q). The steady-state armature magnetic field is shown in Fig. 8.

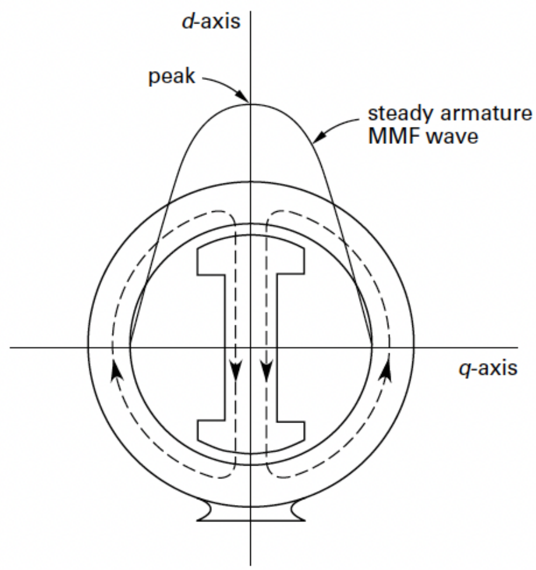


Figure 8: Steady-State Armature Magnetic Field

The reaction that occurs between the magnetic field generated by the rotor being turned and the armature reaction on the stator is shown in Fig. 9 for both a generator and a motor. This reaction is a function of Faraday's law.[11] Note the torque angle, which results from the interaction of the magnetic fields.

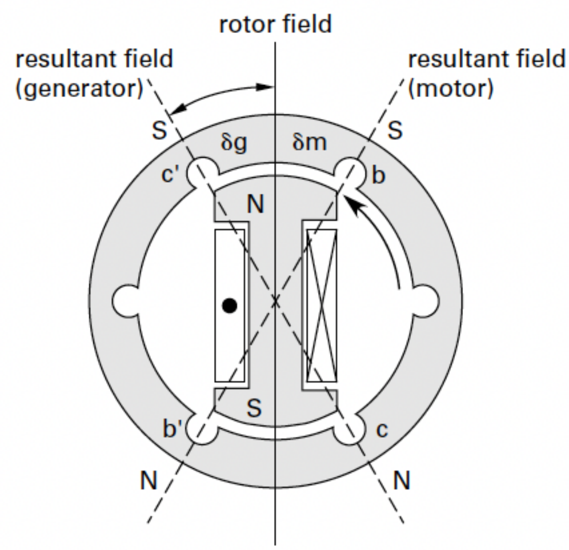


Figure 9: Magnetic Field Generated by Synchronous Generator

The direct-axis synchronous reactance can be measured using open-circuit and short-circuit tests. This is an unsaturated value and neglects small stator resistances.

Equation 30: Direct Axis Synchronous Reactance

$$X_{ds} = \frac{V_{oc}}{I_{sc}}$$

The unsaturated short-circuit ratio (SCR) is the inverse of the direct-axis synchronous reactance, and in per-unit terms is

Equation 31: Short Circuit Ratio

$$\mathrm{SCR} = \frac{1}{X_{ds(pu)}}$$

[11] In short, if generating a flux, an opposing flux will be generated in return.

The total power output of a synchronous generator on a three-phase basis depends on the parameters mentioned and the physical construction of the generator, as shown in Fig. 10. Salient poles tend to concentrate the flux more readily, but cylindrical rotors can withstand greater speeds.

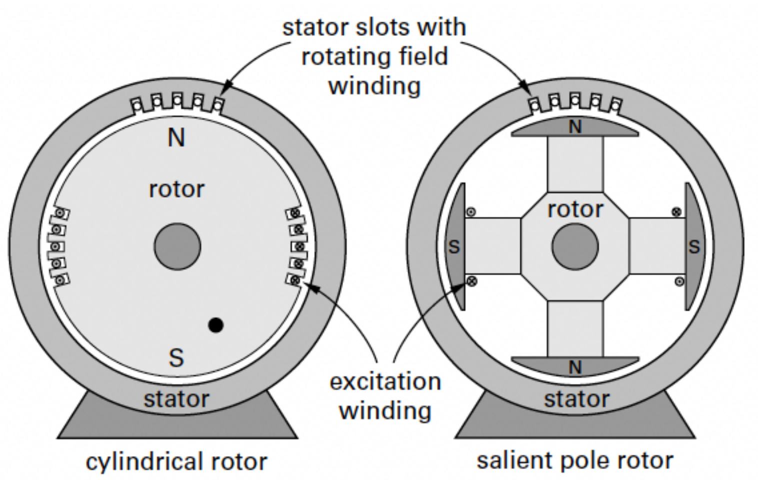


Figure 10: Synchronous Generator Construction

For cylindrical rotor, the steady power is

Equation 32: Cylindrical Rotor Steady-State Power

$$P_e = \frac{3E_0E}{X_s}\sin\delta$$

The torque is obtained by dividing by the synchronous speed.

Equation 33: Synchronous Machine Torque

$$T_e = \frac{3E_0E}{X_s n_s}\sin\delta$$

For a salient power rotor, the steady state power is more complex and involves the two reactances.

Equation 34: Salient Pole Steady-State Power

$$P_e = \frac{3E_0E}{X_s}\sin\delta + \frac{3E^2\left(X_d - X_q\right)}{2X_dX_q}\sin 2\delta$$

Again, the torque is obtained by dividing by the synchronous speed.

Equation 35: Salient Pole Power

$$P_e = \frac{3E_0E}{X_s n_s}\sin\delta + \frac{3E^2\left(X_d - X_q\right)}{2X_d X_q n_s}\sin 2\delta$$

The *synchronizing power* is the varying of the power relative to the torque angle. The synchronizing power is also called the *stiffness of coupling*, *stability*, or *rigidity factor*. It is found by differentiating Eq. 32 and Eq. 34 with respect to the torque angle. The results follow.

Equation 36: Synchronizing Power Cylindrical Rotor

$$P_{\text{sync}} = \frac{3E_0E}{X_s}\cos\delta$$

Equation 37: Synchronizing Power Salient Pole Rotor

$$P_{\text{sync}} = \frac{3E_0E}{X_s}\cos\delta + \frac{3E^2\left(X_d - X_q\right)}{X_d X_q}\cos 2\delta$$

Synchronous Motor/Reactive Generator

The vector relationship for a single phase of a synchronous machine is shown in Fig. 11. V_t is the terminal voltage. When functioning as a motor, the voltage is supplied by an external source. When functioning as a generator, the voltage is supplied by the synchronous generator itself. V_0 is the generated voltage from the synchronous machine (as a motor or generator), shown as E earlier.[12] The term I_a is the armature current flowing in the stator. The term jI_aX_s is the inductive voltage resulting from the coils of the armature reactance, 90° offset from the armature current.

In Fig. 11(a), the motor is operating as a lagging power factor (pf) motor load, the underexcited case. The excitation level is represented by V_0 and is controlled by a resistor in series with the rotating field. A synchronous motor turns at the speed of the rotating magnetic field on the stator. The rotor lags by the torque angle but turns at the synchronous speed. The strength of the field on the rotor interacting with the stator magnetic field is what maintains the torque and subsequently the speed. If the excitation is lowered excessively, the magnetic field may not be able to remain synchronized with the stator depending on the motor load. If this occurs, the machine may suffer

[12] In other electrical books, a generated voltage V is shown as E.

pole slip. Pole slippage results in excessively high induced voltages and damaging currents that may render the motor useless.

In Fig. 33.11(b), the excitation level increases when the V_0 phasor increases. As long as the motor load has not increased, the excitation voltage phasor follows the line of constant power. The torque angle becomes smaller in magnitude due to the increased strength of the magnetic field on the rotor, but the overall power delivered is determined by the load, and therefore remains constant. At this point, the power factor angle of the motor is zero, which is also called the unity power *factor*.

In Fig. 11(c), the machine is operating as a motor with a leading power factor and excitation has increased to the point where the armature current causes the terminal voltage, as opposed to the original system source causing the terminal voltage. The motor then appears to the supplying system as a *capacitive load*, or a *reactive generator*. Therefore, a synchronous motor can correct the power factor of the supplying distribution system by reducing the power factor angle to zero (if the motor is capable of supplying the entire reactive load on the system). This will cause the total loading on the system source to appear resistive, minimizing the total current.[13] This occurs because the reactive current component, previously supplied by the system source, is now supplied by the motor, which is essentially acting as a large capacitor.

[13] Correcting the power factor means bringing it to zero so that the total loading appears resistive and the total current is at a minimum.

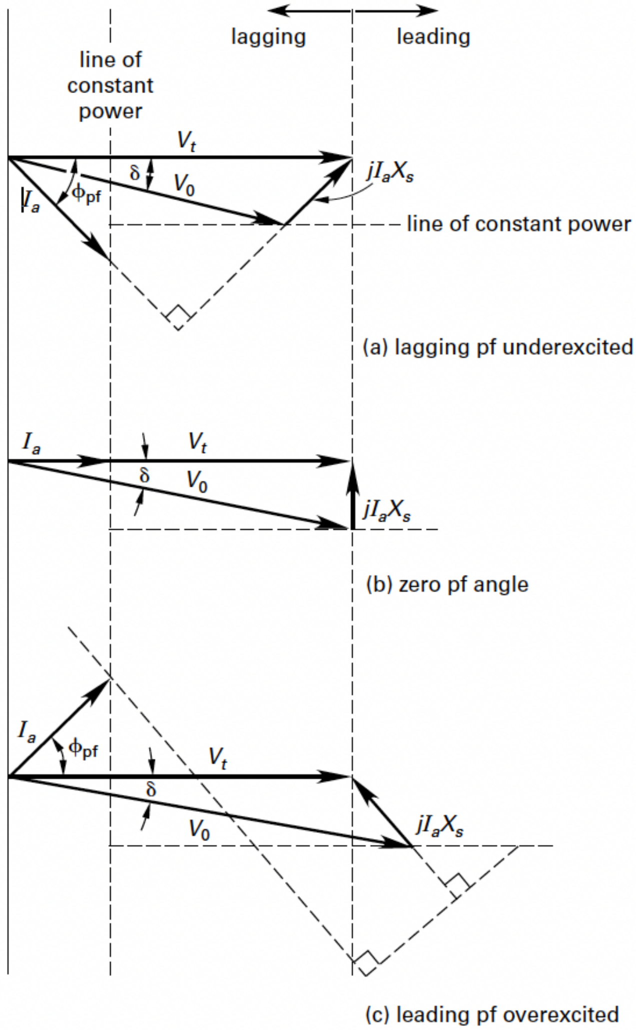


Figure 11: Synchronous Motor Phase Relationship

Induction Motors

Induction motors are essentially constant-speed devices that receive power through induction—there are no brushes or slip rings. A motor can be considered as a rotating transformer secondary (the rotor) with a stationary primary (the stator). The stator field rotates at the synchronous speed given in Eq. 20. An emf is induced as the stator field moves past the rotor conductors. Because the rotor windings have reactance, the rotor field lags the induced emf.

In order to have a change in flux linkage, the rotor must turn at less than the synchronous speed. The difference in speed is small but essential. *Percent slip*, *s*, typically 2% to 5%, is the percentage difference in speed between the rotor and stator fields. Slip can be expressed as either a decimal or a fraction (e.g., 0.05 slip or 5% slip). Percent slip and *percent synchronism* are complements (i.e., add to 100%). Slip in rpm is the difference between actual and synchronous speeds.

Slip in rpm is the difference between actual and synchronous speeds, and is often given as a percentage. The various equations for slip follow.

Equation 38: Slip

$$s = \frac{n_s - n}{n_s}$$

Equation 39: Slip Percentage

$$s_{\%} = \frac{n_s - n}{n_s} \times 100\%$$

Equation 40: Slip Rotational

$$s = \frac{\omega_{\text{mech,s}} - \omega_{\text{mech}}}{\omega_{\text{mech,s}}}$$

The stator is identical to that in an alternator or synchronous motor. In a *wound rotor*, the rotor is similar to an armature winding in a dynamo.[14] However, in *a squirrel-cage rotor* there are no wire windings at all.[15] The squirrel-cage rotor consists of copper or aluminum bars embedded in slots in the cylindrical iron core of the rotor. The ends of the bars are shorted by conductive rings, as illustrated in Fig. 12, to form loops. The rotor and its components, connections, and lines constitute the rotor circuit or secondary circuit.

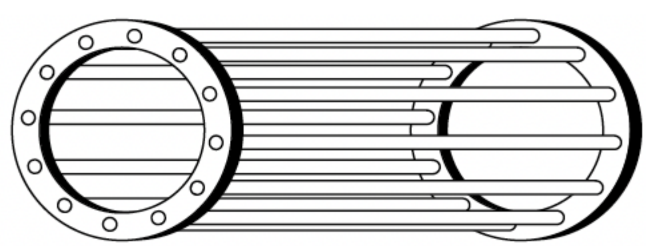


Figure 12: Squirrel-Cage Rotor Induction Motor

[14] Windings can be incorporated in the rotor to obtain better speed control or high torque.
[15] If the slip rings of a wound rotor are shorted, the motor behaves like a squirrel-cage motor.

Example 4

An induction motor developing 10 hp is connected to a three-phase, 240 V (rms), 60 Hz power line. The stator windings are connected in a wye configuration. The synchronous speed is 1800 rpm, but the motor turns at 1738 rpm when loaded. Its energy efficiency is 80% and power factor (pf) is 70%. Calculate the (a) slip, (b) number of poles, (c) line current drawn, and (d) phase voltage.

Solution

(a) From Eq. 39,

$$s = \frac{\omega_{\text{mech,s}} - \omega_{\text{mech}}}{\omega_{\text{mech,s}}} = \frac{1800\text{ rpm} - 1738\text{ rpm}}{1800\text{ rpm}} = 0.03444$$

(b) The number of poles is calculated from the synchronous speed.

$$p = \frac{120f}{n_s} = \frac{(120)(60\text{ Hz})}{1800\ \frac{\text{rev}}{\text{min}}} = 4$$

(c) The total real power input is found from the horsepower. One horsepower is equal to 745.7 W.

$$P = \frac{P}{3} = \frac{9321\text{ W}}{3} = 3107\text{ W}$$

Because the phase and line currents are identical in wye-connected loads, the line current is

$$P = I_p V_p(\text{pf})$$

$$I_p = \frac{P_p}{V_P(\text{pf})} = \frac{3107\text{ W}}{\left(\frac{240\text{ V}}{\sqrt{3}}\right)(0.70)} = 32.03\text{ A}$$

(d) The voltage across each phase is

$$V_p = \frac{V}{\sqrt{3}} = \frac{240\text{ V}}{\sqrt{3}} = 138.6\text{ V}$$

Induction Motor Equivalent Circuit

Figure 13(a) illustrates the equivalent circuit for an induction motor.[16] It is very similar to the equivalent circuit for a transformer. R_1 and X_1 represent the equivalent stator resistance and reactance, respectively. G_{nl} and B_{nl} represent the stator core loss and susceptance, respectively, as determined from no-load testing. The dashed line represents the air gap across which energy is transferred to the rotor. R_2 and X_2 are the equivalent rotor resistance and reactance, respectively. There is no element to model the rotor core loss, which is negligible. Rotational losses are included in the stator core loss element, G. Any equivalent load resistance is included in the rotor resistance. The ratio of transformation, a, is taken as 1.0 for a squirrel-cage motor.[17]

Using an adjusted voltage, V_{adj}, simplifies the model, as shown in Fig. 13(b). The relationship between the applied terminal voltage, V_1, and the adjusted voltage is

Equation 41: Induction Motor Adjusted Voltage

$$V_{\text{adj}} = V_1 - I_{\text{nl}}\left(R_1 + jX_1\right)$$

Equation 42: Induction Motor Adjusted Voltage Approximation

$$V_{\text{adj}} \approx V_1 - I_{\text{nl}}\sqrt{R_1^2 + jX_1^2}$$

[16] This equivalent circuit cannot be used for a double squirrel-cage motor.
[17] With a phase-wound rotor, the brushes can be lifted from the slip rings. The ratio of transformation, a, can be determined as the ratio of applied voltage to voltage across the slip rings. This cannot be done with a squirrel-cage motor, so the equivalent circuit parameters are redefined with the ratio of transformation.

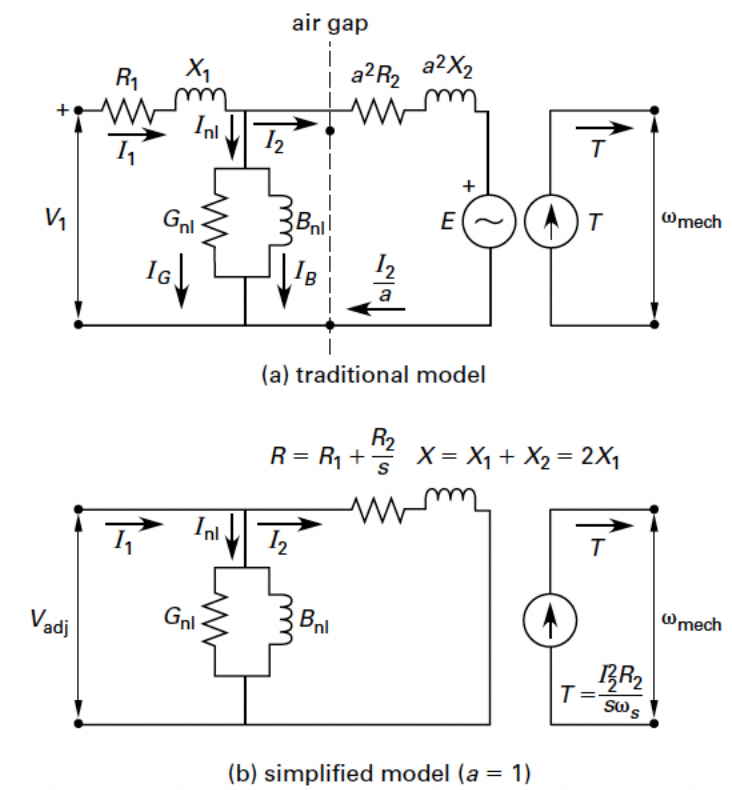


Figure 13: Equivalent Circuits of an Induction Motor

The total series resistance per phase, R, is

Equation 43: Induction Motor Phase Resistance

$$R = R_1 + \frac{R_2}{s}$$

Equation 44 gives the torque-speed relationship predicted by this model.

Equation 44: Induction Motor Torque vs Speed

$$T_p = \frac{I_2^2 R_2}{s\omega_s} = \frac{V_{adj}^2 R_2}{s\omega_s\left(\left(R_1 + \frac{R_2}{s}\right)^2 + X^2\right)}$$

Equation 45: Induction Motor Total Torque

$$T_t = 3T_p$$

Equation 46 is used to find the rotor air gap power. The subscript 2 indicates the rotor resistance referred to the stator, much as in a transformer.

Equation 46: Inductor Motor Rotor Power

$$P_r = 3I_2^2 \frac{R_2}{s}$$

Rotor copper losses are sP_r, so the mechanical output power is

Equation 47: Induction Motor Mechanical Power

$$P_m = \left(1 - s\right) P_r$$

The gross mechanical torque, neglecting windage and friction, is a follows. The term p is the number of poles.

Equation 48: Induction Motor Torque

$$T_m = \frac{P_m}{\left(1-s\right)\omega_s} = \frac{pP_r}{4\pi f}$$

Operating Characteristics of Induction Motors

Under normal operating conditions, slip is small (less than 0.05) and Eq. 49 predicts that the torque will be directly proportional to slip, s, and inversely proportional to the rotor resistance, R_2. At low speeds, the reactive term is larger than the resistive term in Eq. 50.

Equation 49: Induction Motor Torque, High Speed

$$T_p \approx \frac{V_{\text{adj}}^2 s}{\omega_s R_2} \quad \text{[high speed, per phase]}$$

Equation 50: Induction Motor Torque, Low Speed

$$T_p \approx \frac{V_{\text{adj}}^2 R_2}{s\omega_s R_2} \quad \text{[low speed, per phase]}$$

The starting torque per phase is proportional to rotor winding resistance and terminal voltage and can be found by setting s equal to one in Eq. 50.

Equation 51: Induction Motor Starting Torque

$$T_{\text{starting}} \square \frac{V_{\text{adj}}^2 R_2}{\omega_s R_2} \quad \text{[starting, per phase]}$$

The maximum torque (known as *breakdown torque*) is independent of the rotor circuit resistance, but the rotor resistance does affect the speed at which the maximum torque occurs. A large rotor circuit resistance only causes the maximum torque to occur at a larger slip. Maximum torque varies directly with the square of the stator voltage.

Equation 52: Induction Motor Maximum Torque

$$T_{\max} = \frac{V_{\text{adj}}^2}{2\omega_s\left(R_1 + \sqrt{R_1^2 + X^2}\right)} \quad \text{[per phase]}$$

Equation 53: Induction Motor Slip at Max Torque

$$s_{T,\max} = \frac{R_2}{\sqrt{R_1^2 + X^2}}$$

Figure 14 illustrates typical characteristic curves of an induction motor. Such curves are conventionally provided for the motor running at its optimum efficiency and power factor. Curves can be provided for polyphase or single-phase operation.

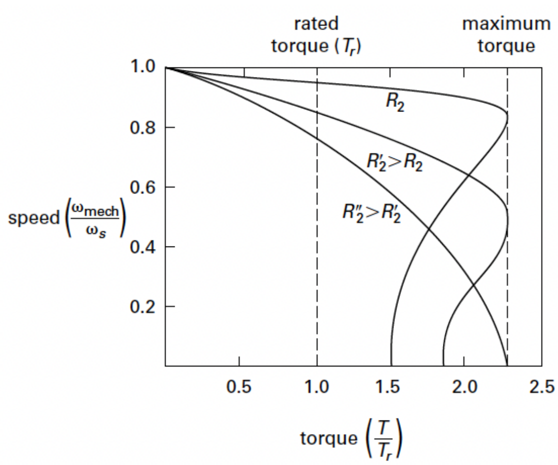


Figure 14: Induction Motor Characteristic Curves

Testing Induction Motors

Performance of induction motors is evaluated in a manner analogous to transformer testing. The *no-load motor test* (*running-light test*) corresponds to an open-circuit transformer test. This test determines the values of B and G in the equivalent circuit. The *blocked-rotor test* corresponds to a closed-circuit transformer test and determines the values of R and $\mathbf{X}$.

In a *no-load test*, the motor is run at the rated voltage without load. The line voltage, V, line current, I_{nl}, and power per phase, P_{nl}, are measured.

Equation 54: No-Load Power Factor

$$\text{pf}_{\text{nl}} = \cos\theta_{\text{nl}} = \frac{P_{\text{nl}}}{V_{\text{adj}} I_{\text{nl}}}$$

Referring to Fig. 13, the magnetization (quadrature) current, I_B, and the in-phase component of the exciting current, I_G, are

Equation 55: Magnetization/Quadrature Current

$$I_B = I_{\text{nl}} \sin\theta_{\text{nl}}$$

Equation 56: Exciting Current

$$I_G = I_{\text{nl}} \cos\theta_{\text{nl}}$$

The stator parameters are as follows.

Equation 57: Stator Conductance

$$G_{\text{nl}} = \frac{P_{\text{nl}}}{V_{\text{adj}}^2}$$

Equation 58: Stator Susceptance

$$B_{\text{nl}} = -\sqrt{Y_{\text{nl}}^2 - G_{\text{nl}}^2} = -\sqrt{\left(\frac{I_{\text{nl}}}{V_{\text{nl}}}\right)^2 - G_{\text{nl}}^2} \approx \frac{I_B}{V_{\text{adj}}}$$

In a *blocked-rotor test*, the rotor is blocked. A (low) voltage is applied and measured so that the rated current flows. The phase current, I_b, power per phase, P_b, and phase voltage, V_b, are measured.

The phase current lags the phase voltage by the following angle.

Equation 59: Block Rotor Phase Current/Voltage

$$\cos\theta_b = \frac{P_b}{I_b V_b}$$

The stator reactance, X_1, is usually assumed to be one-half of X, the remainder being the rotor reactance. However, because performance depends greatly on R_2, the division of R must be less arbitrary. One approach is to measure the DC resistance between the terminals and use this as an approximate value of $2R_1$.[18] (This assumes wye-connected motor windings.) The remainder of R is given to R_2.

Equation 60: Induction Motor Reactance

$$X = X_1 + X_2 = 2X_1 = \frac{V_b}{I_b \sin\theta_b} = \sqrt{Z^2 - R^2}$$

Equation 61: Induction Motor Resistance

$$R = R_1 + R_2 = \frac{P_B}{I_b^2}$$

Equation 62: Induction Motor Impedance

$$Z = \frac{V_b}{I_b}$$

Starting Induction Motors

Induction motors draw their maximum current when starting (i.e., when slip, s, is one). The starting torque for a polyphase motor varies directly with the square of the stator voltage but also depends on the rotor resistance and reactance. (Starting torque for a single-phase induction motor is zero. Therefore, single-phase induction motors must be brought up to speed by some other means.[19]) For a given stator voltage, there is a particular rotor resistance that maximizes starting torque.

[18] This value is corrected empirically, but the correction is beyond the scope.
[19] Polyphase induction motors will run but will not start on a single phase. There is a danger of current overload if synchronization is lost during single-phase operation. Protective elements (e.g., circuit breakers) are used to limit current.

Increasing or decreasing resistance from this optimum value decreases the starting torque. Starting current can be calculated from Eq. 63 by setting *s* equal to one.

Equation 63: Induction Motor Starting Current

$$I_{\text{starting}} = I_1 = I_{\text{nl}} + I_2 = I_{\text{nl}} + \frac{V_{\text{adj}}}{R_1 + \frac{R_2}{s} + jX}$$

The *blocked*- (*locked*-) *rotor current* is the current drawn by the motor with the rotor held stationary. It is a worst-case starting current because the rotor begins to move immediately upon starting, reducing the current drawn. Free-rotor starting current is approximately 75% of the blocked-rotor current.

Speed Control for Induction Motors

The rotational speed of an induction motor depends on the number of poles, line voltage, supply frequency, and rotor circuit resistance. For a given machine, the number of poles cannot be varied without excessive complexity in winding, switching, and increased manufacturing cost. Because the breakdown torque is proportional to the square of the voltage, reducing the voltage may stall the motor. As a result, voltage speed control is rarely used. Changing the supply frequency is also impractical in most instances. Introducing a resistance in series with the rotor decreases the motor speed but is possible only in wound-rotor motors.

Speed control is commonly accomplished by introducing a foreign voltage in the secondary (rotor) circuit. If the foreign voltage opposes the voltage induced in the secondary circuit, the motor speed will be reduced, and vice versa.

If two induction motors are available, one can be used to control the other by connecting them in *cascade*. The shafts are rigidly connected and the rotor and stator windings are interconnected. The two armatures can also be constructed on a single shaft.

Power Transfer in Induction Motors

Figure 15 illustrates the power transfer in an induction motor. The equations that follow [Eqs. 64-69] define the values on a per phase basis.

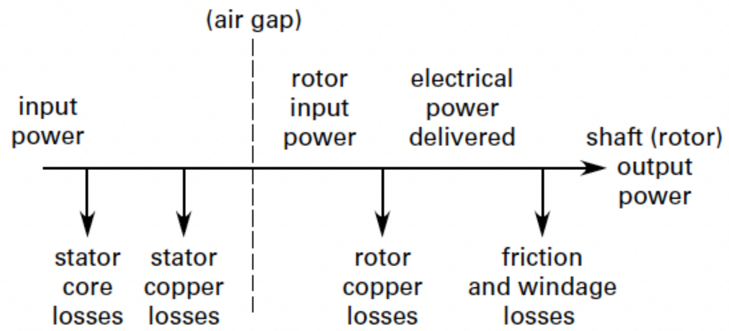


Figure 15: Induction Motor Power Transfer

Equation 64; Input Induction Motor Power

$$P_{\text{input}} = V_1 I_1 \cos\theta$$

Equation 65: Stator Copper Losses

$$L_{\text{stator}} = I_1^2 R_1$$

Equation 66: Input Rotor Power

$$P_{\text{input,rotor}} = \frac{I_2^2 R_2}{s}$$

Equation 67: Rotor Copper Losses

$$L_{\text{rotor}} = I_2^2 R_2$$

Equation 68: Electric Power Delivered

$$P_{\text{delivered,electrical}} = I_2^2 R_2 \left(\frac{1-s}{s} \right)$$

Equation 69: Mechanical-Shaft Power

$$P_{\text{output,mech}} = T\omega_{\text{mech}}$$

Core losses are constant [see Power Losses]. Copper losses are proportional to the square of the delivered power.

Example 5

A 15 hp induction motor with six poles operates at 80% efficiency on a three-phase, 240 V (rms), 60 Hz line. The following losses are observed for full-load operation.

Stator Copper Loss:	540 W
Friction/Windage Loss:	975 W
Core Loss:	675 W

What are the (a) speed and (b) torque when the motor delivers half power?

Solution

(a) The full-load output power is

$$P = (15 \text{ hp})\left(745.7 \ \frac{\text{W}}{\text{hp}}\right) = 11{,}186 \text{ W}$$

The input power is

$$\begin{aligned} P_{\text{in}} &= \frac{P_{\text{out}}}{\eta} \\ &= \frac{11{,}186 \text{ W}}{0.80} \\ &= 13{,}982 \text{ W} \end{aligned}$$

The full-load rotor copper loss is

$$\begin{aligned} \text{rotor copper loss} &= 13{,}982 \text{ W} - 11{,}186 \text{ W} \\ &\quad -540 \text{ W} - 975 \text{ W} - 675 \text{ W} \\ &= 606 \text{ W} \qquad \text{[full power]} \end{aligned}$$

The output power at half-load is

$$P = \left(\frac{1}{2}\right)(15 \text{ hp})\left(745.7 \ \frac{\text{W}}{\text{hp}}\right) = 5593 \text{ W}$$

At half load, the friction and core losses are unchanged. From Eq. 67 and Eq. 68, the ratio of actual to full-load copper losses is equal to the square of the ratio of actual to full-load output power.

$$\text{stator copper loss} = \left(\frac{1}{2}\right)^2 (540 \text{ W}) = 135 \text{ W}$$

$$\text{rotor copper loss} = \left(\frac{1}{2}\right)^2 (606 \text{ W}) = 152 \text{ W}$$

The rotor input power is

$$P_{\text{in}} = 5593 \text{ W} + 975 \text{ W} + 152 \text{ W} = 6720 \text{ W}$$

The synchronous speed is

$$n_s = \frac{120f}{p} = \frac{(120)(60 \text{ Hz})}{6} = 1200 \text{ rpm}$$

The slip at half-power is found from Eq. 66 and Eq. 67, see Fig. 15.

$$s = \frac{\text{rotor copper loss}}{\text{rotor input power}} = \frac{I_2^2 R_2}{\frac{I_2^2 R_2}{s}} = s$$

$$= \frac{152 \text{ W}}{6720 \text{ W}} = 0.0226$$

The speed at half-load is

$$\begin{aligned} n &= n_s(1-s) \\ &= \left(1200 \frac{\text{rev}}{\text{min}}\right)(1 - 0.0226) \\ &= 1173 \text{ rev/min} \end{aligned}$$

(b) The torque is found from Eq. 5

$$T_{\text{ft-lbf}} = \frac{5252 P_{\text{horsepower}}}{\eta_{\text{rpm}}} = \frac{5252\left(\frac{1}{2}\right)(15 \text{ Hp})}{1173 \frac{\text{rev}}{\text{min}}} = 33.6 \text{ ft-lbf}$$

Single-Phase Motors

A single-phase motor works similarly to a three-phase induction motor. The stator carries a main winding and a north-south pole set. An auxiliary winding, with the same number of poles, usually operates during the brief period of motor startup and is used to set the rotor in the desired direction. Once the direction of the rotor rotation is established, the auxiliary winding can be removed and the motor will continue to operate.[20,21]

Split-Phase Motors

A *resistance split-phase motor*, usually called a *split-phase motor*, is shown in Fig. 16. The stator winding and the auxiliary winding establish magnetic fields 90° apart. Because the auxiliary winding is made from finer wire than the stator winding (main winding), the auxiliary winding has a higher resistance and lower reactance. As a result, the main winding current lags the auxiliary winding current. For the connections shown, the result is in a counterclockwise rotation. To change the direction of rotation, reverse the connection of either the auxiliary winding (i.e., the start winding) or the main winding (i.e., the run winding), but not both. The corresponding phasor diagram is shown in Fig. 17. The line current at startup is between 6 to 7 times the normal running current. This motor type is used where infrequent starting of low or moderate torque loads is required and a low cost is imperative.

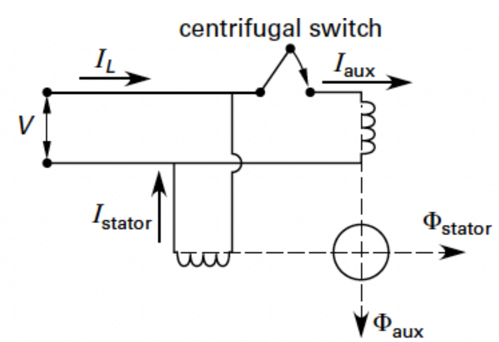


Figure 16: Split-Phase Motor Magnetic Fields

[20] Consider the single-phase motor as a two-phase motor at startup. Once started, the field of the rotor interacts with the stator (main) field to maintain the revolving magnetic field. This revolving field should more accurately be termed an alternating field.

[21] Two theories are used to describe the complex principle of operation, the double revolving field theory and the cross-field theory. Both result in a revolving magnetic field that sustains itself once the rotor is in motion. The theories are involved.

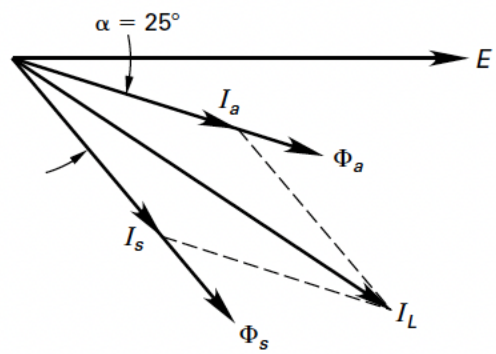


Figure 17: Split-Phase Motor Phasor Diagram

Capacitor-Start Motors

A *capacitor-start motor* is shown in Fig. 18. The stator and auxiliary windings establish magnetic fields 90° from one another. The auxiliary winding is made of finer wire than the stator (i.e., main) winding. The main difference between the capacitor-start and split-phase motor is that the auxiliary windings of the capacitor-start motor have nearly as many turns as the main windings. The auxiliary winding current leads the stator current by approximately 80°. This allows for greater starting torque than the split-phase design (for the same torque, less current flows in the auxiliary winding). Therefore, the capacitor-start motor for a given output generates less heat. For the connections in Fig. 17(a), the rotation is counterclockwise. To change the direction of rotation, reverse the connection of either the auxiliary winding (i.e., start winding) or the main winding (i.e., run winding), but not both. The corresponding phasor diagram is shown in Fig. 19. The line current at startup is 4 to 5 times the nominal running current. This motor type is used where high starting torques are required.

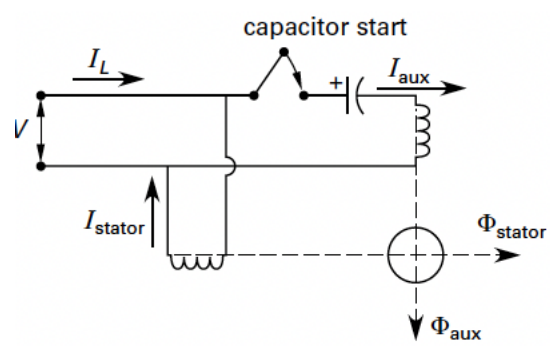


Figure 18: Capacitor-Start Motor Magnetic Fields

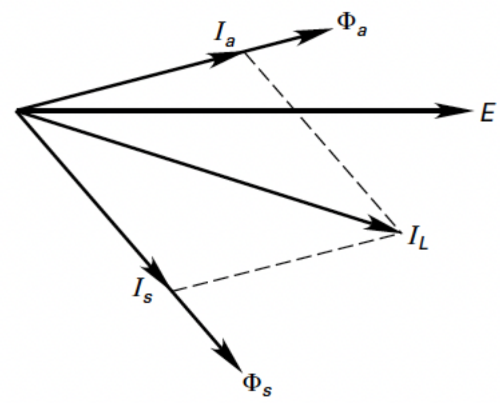


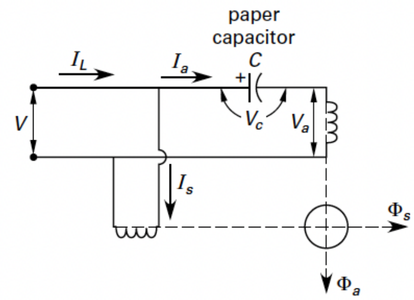
Figure 19: Capacitor Start Motor Phasor Diagram

Single-Phase Motor Vibrations

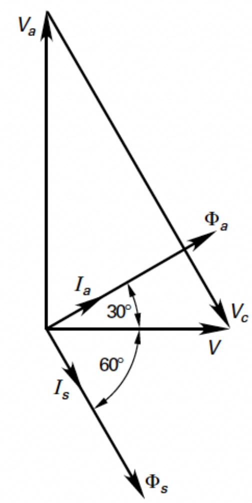
A single-phase motor receives pulsating AC power even while delivering constant mechanical power that causes vibration. This vibration occurs at twice the line frequency. In contrast, two-phase and three-phase motors do not vibrate because the total instantaneous power received from all the phases remains constant (for a constant load).

Capacitor-Run Motors

A *capacitor-run motor*, also called *capacitor-start capacitor-run*, is shown in Fig. 20(a). The stator and auxiliary windings establish magnetic fields 90° from one another. The auxiliary winding is made of finer wire than the stator (i.e., main) winding. The primary difference between the capacitor-run and capacitor-start motors is that the auxiliary windings of the capacitor-run motor have more turns than the main windings. The auxiliary winding current leads the stator current by 90° at full load. This allows for a higher power factor and increased efficiency but a lower starting torque. For the connections shown in Fig.20(a), the rotation is counterclockwise. To change the direction of rotation, reverse the connection of either the auxiliary winding (i.e., start winding) or the main winding (i.e., run winding), but not both. The corresponding phasor diagram is shown in Fig. 20(b). The line current at startup is 4 to 5 times the nominal running current. This motor type is used where quiet operation is desired (e.g., hospitals or recording studios).



(a) startup magnetic fields



(b) phasor diagram at full load

Figure 20: Capacitor-Run Motor

Shaded-Pole Motors

A *shaded-pole motor* is shown in Fig. 21. The main winding creates the driving magnetic flux. Some of the flux links the copper ring on pole A, some links the copper ring on pole B, and the remainder links the rotor. The copper rings have low resistance; therefore, higher currents flow (with resulting stronger magnetic fields) in the rings as opposed to the main stator structure. The flux in the copper rings lags the originating fluxes, which establishes a weak revolving magnetic field. In the figure, the direction of rotation is from the unshaded region to the shaded region (copper ring side) of the pole. This motor type has low starting torque, poor efficiency, and a low power factor. However, the simple and inexpensive construction makes it useful in very low power applications.

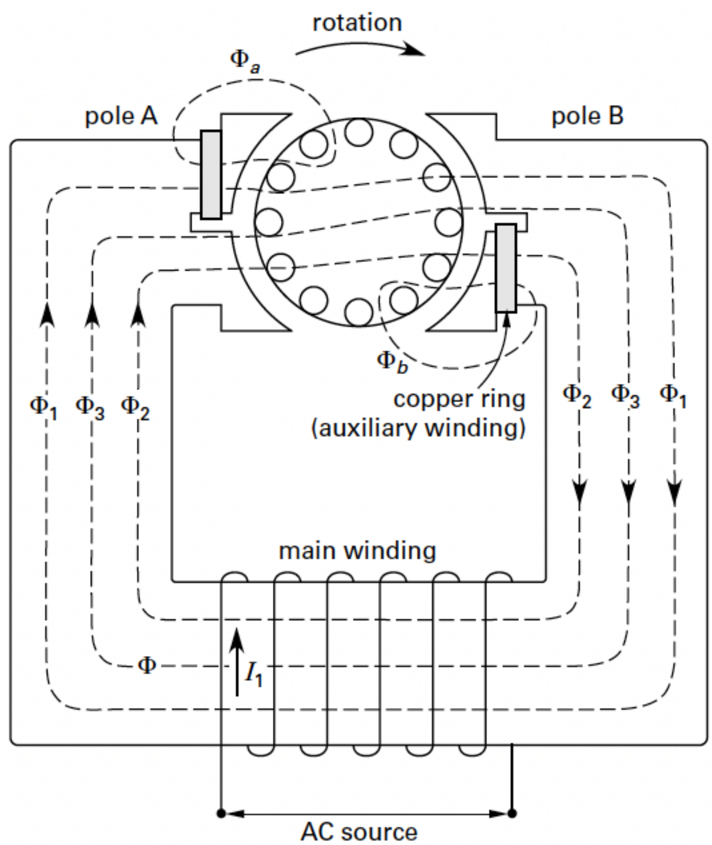


Figure 21: Shaded-Pole Motor

Universal Motors

A *universal motor*, also called a *series motor*, is shown in Fig. 22. This is an alternating current series motor whose construction is very similar to a DC series motor. The primary difference is that the magnetic circuit in the stator is completely laminated to decrease eddy current losses. The motor can operate on AC or DC power. When operating on AC power, the magnetic fields on the stator and the rotor change with each alteration of the AC signal, thereby maintaining the relative relationship between the poles. As a result, the direction of rotation remains the same as the alternating current changes from positive to negative. Because the armature current and the flux reverse simultaneously, no revolving magnetic field exists. Therefore, the motor operates using the same principle as a DC series motor. Universal motors tend to be noisy, have relatively short lives, operate at high speeds, and have a high starting torque. They are used in vacuum cleaners, portable tools, and toys. Larger versions are used in locomotive engines.

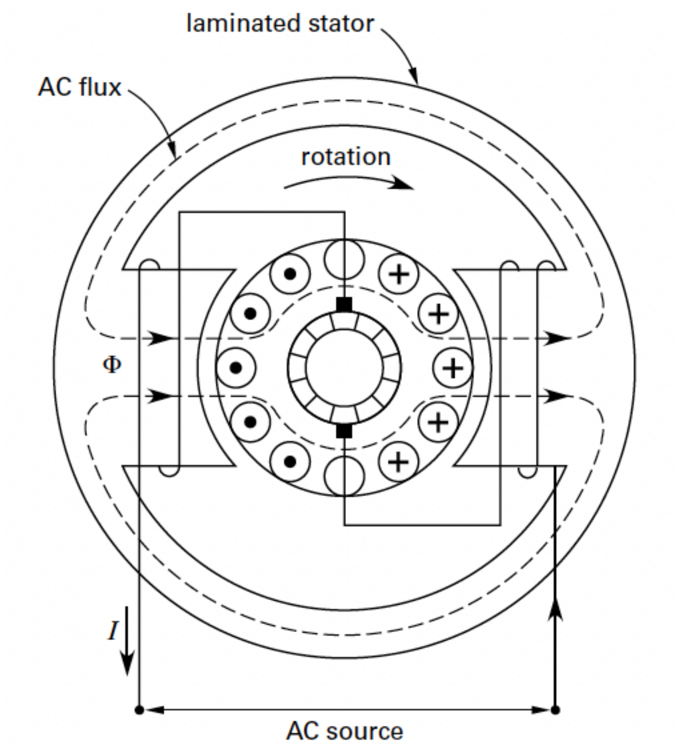


Figure 22: Universal Motor

Hysteresis Motors

A *hysteresis motor* is a synchronous motor without poles or direct-current excitation. One phase of a hysteresis motor is shown in Fig. 23. The revolving magnetic field on the stator is from two-phase (i.e., main and auxiliary windings) or three-phase windings. The rotor is a ceramic permanent magnet with resistivity near that of an insulator. Therefore, it is impossible to set up eddy currents in the rotor. As the magnetic field of the stator rotates, it produces poles of opposite polarity on the rotor that are attracted to the revolving poles causing a rotation. Domains are continuously created in the rotor.

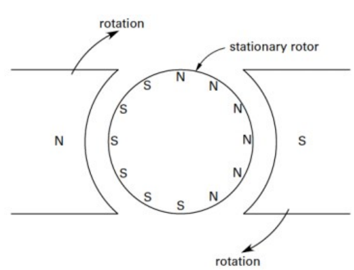


Figure 23: Hysteresis Motor

During each cycle of the stator, the domains on the rotor complete a cycle called a *hysteresis loop*. The area of the loops represents losses. These hysteresis losses are evident as heat in the rotor. The mechanical power driving the rotor comes from power used to drive the domains on the rotor, and is equal to the hysteresis losses. The resulting torque, T, is given by the energy dissipated in the rotor, E_h, in joules per revolution, which is divided by a constant that performs the necessary unit conversions.

Equation 70: Hysteresis Motor Torque

$$T = \frac{E_h}{2\pi}$$

The torque in Eq. 70 is a constant, and therefore, not dependent on the speed of rotation. This is the property that differentiates the hysteresis motor from all others. Additionally, unlike other synchronous motors, which require some specialized means to reach synchronous speed, hysteresis motors generate a nearly constant torque from the start, and therefore, can reach synchronous speed without assistance. (See Fig. 24.)

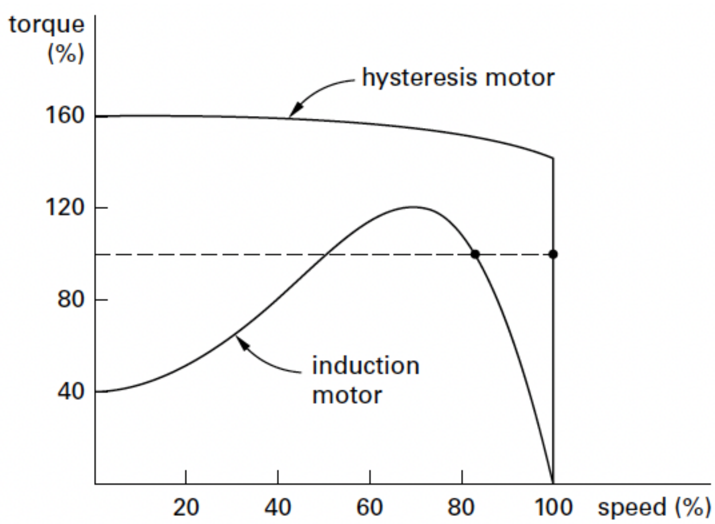


Figure 24: Torque-Speed of Hysteresis Motor

The hysteresis motor is used wherever a constant torque is desired (e.g., in electric clocks, precision audio equipment, and cooling fans).

Reluctance Motors

A *reluctance motor* is shown in Fig. 25. The rotor of a reluctance motor is made of ferromagnetic material and resembles an induction motor, only with salient poles. (See Fig. 26.) The principle of reluctance motors is that lines of magnetic flux are capable of exerting a force to minimize the length of the magnetic circuit (in other words, iron poles carrying a magnetic flux tend to align with one another). In Fig. 25(a), the lines of magnetic flux on the stator pass through a highly permeable rotor. The magnetic field on the stator (in Fig. 25(b)) has rotated clockwise and the lines of magnetic flux are stretched. A force is exerted on the rotor to move it into alignment, as shown in Fig. 25(c). If the stator magnetic field moves rapidly, the rotor operates as a standard induction motor until it reaches near synchronous speed at which point the rotor locks on to the stator field.

A reluctance motor is the most inexpensive synchronous motor type. It can operate at high or low rotating magnetic field speeds. The position of the rotor can be accurately controlled at low speed. It is well suited to variable frequency and electronic speed control mechanisms. The combination of accuracy and speed control allows reluctance motors to be used as the drive motors for control rods in nuclear reactors and other motion control and variable-speed drive applications.

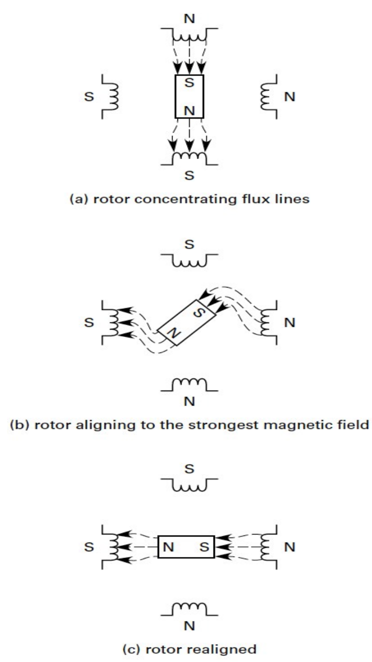


Figure 25: Reluctance Motor

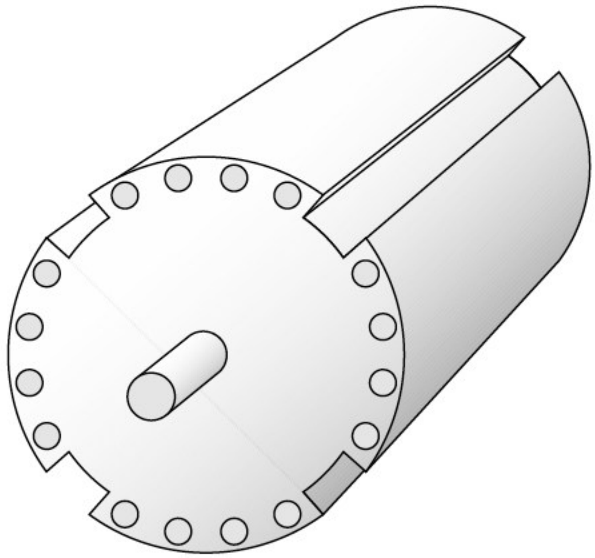


Figure 26: Reluctance Rotor

Speed Control for AC Motors

The speed of an AC motor can be controlled by changing the armature conditions, the field conditions, or both. The type of control is determined by the motor type. *Armature control techniques* include (a) varying the resistance on the rotor side using slip-rings with adjustable resistors attached, (b) controlling the voltage on the armature electronically, or (c) using motor-generator setups to provide synchronous machine excitation. *Field control techniques* include (a) changing the number of poles, (b) controlling the voltage on the field, or (c) controlling the frequency.

Changes to the armature or field can be made manually or automatically. Automatic starters generally control the voltage per hertz, or V/Hz ratio, thereby controlling the torque delivered, up to 60 Hz. Above 60 Hz, the V/Hz ratio is adjusted to maintain a constant torque. The range above 60 Hz, where the frequency can no longer be adjusted, is called the *constant power region* or *constant torque region*.

Variable Frequency Drive (VFD)

A *variable speed drive* (VSD) is an electrical device whose speed varies across a considerable range as a function of the load. It is sometimes called an *adjustable speed drive* (ASD). When the frequency is controlled, the device is electronic and is called a *variable frequency drive* (VFD). Such drives change the frequency in order to change the speed of the rotational field, as shown by

Equation 71: Synchronous Speed Control

$$np = 120f$$

Since the number of poles, *p*, is generally fixed, changing the frequency changes the speed of the rotational field, *n*, which varies the overall speed of the motor.

A block diagram of a VFD is shown in Fig. 27. A VFD converts AC input power to a controllable output which varies in frequency and produces a wave that closely approximates a sine wave.

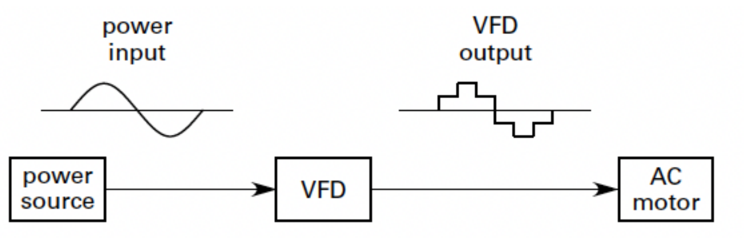


Figure 27: VFD Block Diagram

A simplified electrical schematic of a VFD is shown in Fig. 28. The AC power source is applied to an input rectifier section that changes the signal from AC to DC.[22] In Fig. 28, the VFD shown is a *six-pulse device*, since it generates three pulses on the positive portion of the cycle, and three on the negative portion. This creates a rippled DC signal in the DC bus section.

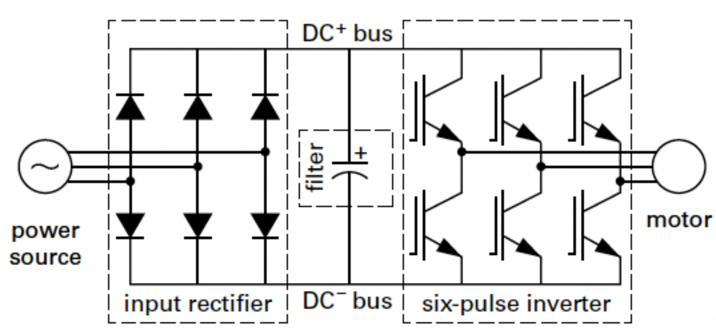


Figure 28: VFD Functional Schematic

Additional pulses may be added by using a phase shifted input and additional rectifier pairs. For example, if the supplying transformer has a delta primary and a delta/wye secondary pair, the wye secondary output is phase-shifted 30° from the corresponding voltage on the delta secondary.

[22] Diodes are shown for simplicity, but silicon-controlled rectifiers (SCRs) or transistors can be used for more precise control.

Using the six outputs from the dual secondary, with two rectifiers for each phase, results in a twelve-pulse device. The greater the number of pulses, the more consistent the DC signal.

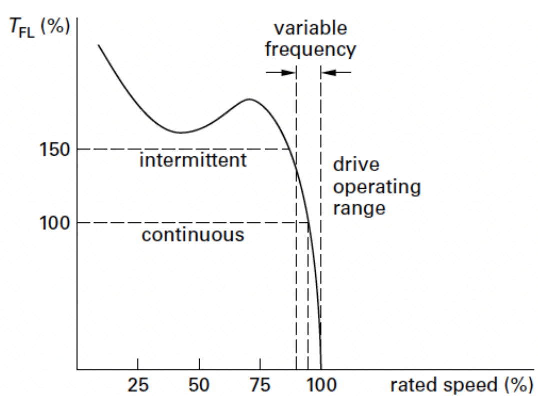
Even without additional pulses to smooth the signal, the DC bus section contains filtering components such as inductors, DC links, chokes, and other items to minimize ripples in the signal. However, the DC bus section's major components are capacitors, only one of which is shown in Fig. 28. The capacitors in the DC bus section store the energy from the pulses, maintaining the DC voltage level and delivering the energy to the inverter section.

The inverter section is controlled by circuitry that controls the firing of the individual transistors. Insulated gate bipolar transistors (IGBTs) are shown. They are common in such circuits because of their high switching speeds and low power consumption in the switching mode. IGBTs use *pulse-width modulation* (PWM), a common method for regulating power supplies, to create an easily modified square wave whose frequency and overall value can be changed. The modifications produce a variable sine wave, as shown in Fig. 27.

VFDs have several benefits compared to other VSDs. They reduce energy consumption, since the speed can be varied to provide only the flow required for a given application. Because the frequency at start-up is low (sometimes called a *soft start*), VFDs also produce lower levels of mechanical stress during start-up than other types of VSDs. Some designs use an active front end, which controls the firing of transistors. This helps minimize harmonics.[23]

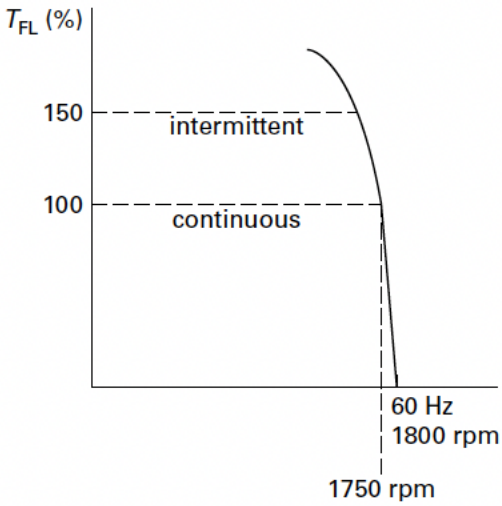
Example 6

An AC induction motor has the torque versus speed characteristic curve shown.



[23] Harmonics that are multiples of 2 are not an issue because they cancel each other out. Third-order harmonics (multiples of 3) also cancel out in a three-phase system. The largest harmonics are the 5th, 7th, 11th, 13th, and so on.

An expansion of the continuous and intermittent operating range is shown, with 1800 rpm as the 100% rated speed.



What is the percent slip and over what frequency range must the variable frequency drive (VFD) operate to control the speed?

Solution

Calculate the slip, s, and the frequency range, Δf.

$$s = \frac{n_s - n}{n_s} = \frac{1800 \text{ rev/min} - 1750 \text{ rev/min}}{1800 \text{ rev/min}}$$
$$= 0.0278 \quad (2.8\%)$$

$$\Delta f = sf_{\text{FL}} = (0.0278)(60 \text{ Hz}) = 1.67 \text{ Hz}$$

Theoretically, the 1.67 Hz difference is all that is required to generate 100% torque. During startup the VFD can be used to limit the frequency of the applied signal to ensure that a surge of current does not occur or is minimal. For example, at approximately 2.5 Hz the motor is producing nearly 150% of rated torque, which should be adequate to start a properly matched load without the surge of five to seven times the full load current normally expected.

Example 7

A 230 V motor designed to operate at 60 Hz is attached to a variable frequency drive (VFD) that maintains a constant V/Hz ratio from 0 Hz to 60 Hz. The motor, though manufactured in the United States, is to be used on a European transmission system operating at 50 Hz.

For proper operation of the motor on the European system, What must the VFD maintain, in terms of V/Hz ratio?

Solution

Maintaining the V/Hz ratio results in constant torque from 0 Hz to 60 Hz. Above this range, the voltage is at a maximum and the field must be weakened, so the V/Hz ratio drops. This is called the flux weakening region or the constant horsepower region. I

In the United States, the V/Hz ratio at rated conditions is

$$R_{\text{V/Hz}} = \frac{230\text{ V}}{60\text{ Hz}} = 3.8\text{ V/Hz}$$

To use the motor in Europe, software programming must change to provide the V/Hz ratio at rated conditions.

$$R_{\text{V/Hz}} = \frac{230\text{ V}}{50\text{ Hz}} = 4.6\text{ V/Hz}$$

REFERENCES

Items (latest editions) in **bold** are highly recommended for in-depth study.

A. **Camara, John A. *PE Power Reference Manual*. Belmont, CA: PPI (Kaplan), 2021.**

B. Earley, Mark, ed. *NFPA 70, National Electrical Code Handbook*. Quincy, Massachusetts: NFPA, 2020.

NOTE

Electrical refers to something related to electricity while "electric" refers to a device or machine that runs on electricity. Nevertheless, the NEC is sometimes referred to as the National Electric Code.

C. IEEE 315-1975. *Graphic Symbols for Electrical and Electronics Diagrams.* New York: IEEE, approved 1975, reaffirmed 1993.

D. IEEE 280-2021. *IEEE Standard Letter Symbols for Quantities Used in Electrical Science and Electrical Engineering.* New York: IEEE.

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F. **Parker, Sybil P., editor in chief. *McGraw-Hill Dictionary of Scientific and Technical Terms,* 5th ed. New York, McGraw-Hill, 1994.**

G. **Wildi, Theodore. *Electrical Power Technology.* New York: John Wiley & Sons, 1981.**

Appendix A: Equivalent Units Of Derived And Common SI Units

Symbol	Equivalent Units			
A	C/s	W/V	V/Ω	$J/(s \cdot V)$
C	$A \cdot s$	J/V	$(N \cdot m)/V$	$V \cdot F$
F	C/V	C^2/J	s/Ω	$(A \cdot s)/V$
F/m	$C/(V \cdot m)$	$C^2/(J \cdot m)$	$C^2/(N \cdot m^2)$	$s/(\Omega \cdot m)$
H	W/A	$(V \cdot s)/A$	$\Omega \cdot s$	$(T \cdot m^2)/A$
Hz	1/s	s^{-1}	cycles/s	$radians/(2\pi \cdot s)$
J	$N \cdot m$	$V \cdot C$	$W \cdot s$	$(kg \cdot m^2)/s^2$
m^2/s^2	J/kg	$(N \cdot m)/kg$	$(V \cdot C)/kg$	$(C \cdot m^2)/(A \cdot s^3)$
N	J/m	$(V \cdot C)/m$	$(W \cdot C)/(A \cdot m)$	$(kg \cdot m)/s^2$
N/A^2	$Wb/(N \cdot m^2)$	$(V \cdot s)/(N \cdot m^2)$	T/N	$1/(A \cdot m)$
Pa	N/m^2	J/m^3	$(W \cdot s)/m^3$	$kg/(m \cdot s^2)$
Ω	V/A	W/A^2	V^2/W	$(kg \cdot m^2)/(A^2 \cdot s^3)$
S	A/V	1/Ω	A^2/W	$(A^2 \cdot s^3)/(kg \cdot m^2)$
T	Wb/m^2	$N/(A \cdot m)$	$(N \cdot s)/(C \cdot m)$	$kg/(A \cdot s^2)$
V	J/C	W/A	C/F	$(kg \cdot m^2)/(A \cdot s^3)$
V/m	N/C	$W/(A \cdot m)$	$J/(A \cdot m \cdot s)$	$(kg \cdot m)/(A \cdot s^3)$
W	J/s	$V \cdot A$	V^2/Ω	$(kg \cdot m^2)/s^3$
Wb	$V \cdot s$	$H \cdot A$	T/m^2	$(kg \cdot m^2)/(A \cdot s^2)$

Appendix B: Physical Constants

Table Note 1

Quantity	Symbol	US Customary	SI Units
Charge			
electron	e		-1.6022×10^{-19} C
proton	p		$+1.6022 \times 10^{-19}$ C
Density			
air [STP][32°F, (0°C)]		0.0805 lbm/ft^3	1.29 kg/m^3
air [70°F, (20°C), 1 atm]		0.0749 lbm/ft^3	1.20 kg/m^3
sea water		64 lbm/ft^3	1025 kg/m^3
water [mean]		62.4 lbm/ft^3	1000 kg/m^3
Distance			
Earth radius[2]	$\oplus$	2.09×10^{7} ft	6.370×10^{6} m
Earth-Moon separation[2]	$\oplus$☾	1.26×10^{9} ft	3.84×10^{8} m
Earth-Sun separtion[2]	$\oplus\odot$	4.89×10^{11} ft	1.49×10^{11} m
Moon radius[2]	☾	5.71×10^{6} ft	1.74×10^{6} m
Sun radius[2]	$\odot$	2.28×10^{9} ft	6.96×10^{8} m
first Bohr radius	a_0	1.736×10^{-10} ft	5.292×10^{-11} m
Gravitational Acceleration			
Earth [mean]	g	32.174 (32.2) ft/sec^2	9.8067 (9.81) m/s^2
Mass			
atomic mass unit	μ or m_μ $\frac{1}{12}m\left({}^{12}C\right)$	3.66×10^{-27} lbm	1.6606×10^{27} kg or 10^{-3} kg mol^{-1} / N_A
Earth[2]	$\oplus$	4.11×10^{23} slugs	6.00×10^{24} kg
Earth [customary U.S.][2]	$\oplus$	1.32×10^{25} lbm	-
Moon[2]	☾	1.623×10^{23} lbm	7.36×10^{22} kg
Sun[2]	$\odot$	4.387×10^{30} lbm	1.99×10^{30} kg
electron rest mass	m_e	2.008×10^{-30} lbm	9.1095×10^{-31} kg
neutron rest mass	m_n	3.693×10^{-27} lbm	1.6750×10^{-27} kg
proton rest mass	m_p	3.688×10^{-27} lbm	1.6727×10^{-27} kg

Pressure			
atmospheric		14.696 (14.7) lbf/in^2	1.0133□10^5 Pa
Temperature			
standard		32°F (492°R)	0°C (273 K)
absolute zero		–459.67°F (0°R)	–273.16°C (0 K)
Velocity[3]			
Earth escape		3.67□10^4 ft/sec	1.12□10^4 m/s
light (vacuum)	c, c_0	9.84□10^8 ft/sec	2.9979 (3.00) □10^8 m/s
sound [air, STP]	a	1090 ft/sec	331 m/s
sound [air, 70°F, (20°C), 1 atm]		1130 ft/sec	344 ft/s
Volume			
Volume: molal ideal gas (STP)[4]		359 ft^3 / lbmol	22.41 m^3/kmol

Table 1 Notes

1. Units come from a variety of sources, but primarily from the Handbook of Chemistry and Physics, The Standard Handbook for Aeronautical and Astronautical Engineers, and the Electrical Engineering Reference Manual for the PE Exam. See also the NIST website at https://pml.nist.gov/cuu/Constants/.
2. Symbols shown for the solar system are those used by NASA. See https://science.nasa.gov/resource/solar-system-symbols/.
3. Velocity technically is a vector. It has direction.
4. The unit "lbmol" is an actual unit, not a misspelling.

Appendix C: Fundamental Constants

Quantity	Symbols	US Customary	SI Units
Avogadro's number	N_A, L		6.022×10^{23} mol^{-1}
Bohr magneton	μ_B		9.2732×10^{-24} J/T
Boltzmann constant	κ	5.65×10^{-24} ft-lbf/ R	1.3805×10^{-23} J/T
electron volt: $\left(\frac{e}{C}\right)$ J	eV		1.602×10^{-19} J
Faraday constant, $N_A e$	F		96485 C/mol
fine structure constant, inverse α^{-1}	α α^{-1}		7.297×10^{-3} (≈1/137) 137.035
gravitational constant	g_c	32.174 lbm-ft/lbf-sec^2	
Newtonian gravitational constant	G	3.44×10^{-8} ft^4 / lbf-sec^4	6.672×10^{-11} N·m^2 / kg^2
nuclear magneton	μ_N		5.050×10^{-27} J/T
permeability of a vacuum	μ_0		1.2566×10^{-6} N/A^2 (H/m)
permittivity of a vacuum, electric constant $1/\mu_0 c^2$	ε_0		8.854×10^{-12} C^2 / N·m^2 (F/m)
Planck's constant	h		6.6256×10^{-34} J·s
Planck's constant: h/2π			1.0546×10^{-34} J·s
Rydberg constant	R_∞		1.097×10^{7} m^{-1}
specific gas constant, air	R	53.3 ft-lbf/lbm- R	287 J/kg·K
Stefan-Boltzmann constant		1.71×10^{-9} BTU/ft^2-hr-°R^4	5.670×10^{-8} W/m^2·K^4
triple point, water		32.02 F, 0.0888 psia	0.01109 C, 0.6123 kPa
universal gas constant	R*	1545 ft-lbf/lbmol- R 1.986 BTU/lbmol- R	8314 J/kmol·K

Table Notes

1. Units come from a variety of sources, but primarily from the Handbook of Chemistry and Physics, The Standard Handbook for Aeronautical and Astronautical Engineers, and the Electrical Engineering Reference Manual for the PE Exam. See also the NIST website at https://pml.nist.gov/cuu/Constants/. The unit in Volume of "lbmol" is an actual unit, not a misspelling.

Appendix D: Mathematical Constants, Signs/Symbols, Maxwell's Equations

Quantity	Symbol	Value
Archimedes' constant (pi)	π	3.1415926536
base of natural logs	e	2.7182818285
Euler's constant	C or τ	0.5772156649

Signs/Symbols	Meaning
$\cdot$	multiplied by
/	divided by
:	ratio
$\gg$	much greater than
$\ll$	much less than
$=$	equals
$\equiv$	identical with
$\sim$	similar to
$\approx$	approximately equals
$\cong$	approximately equals, congruent
$\rightarrow$, $\doteq$	approaches
$\propto$	proportional, varies as
$\therefore$	therefore

Maxwell's Equations

integral form	point form	remarks
$\oint_s \mathbf{D}\cdot d\mathbf{s} = \int_V \rho \, dv$	$\nabla\cdot\mathbf{D} = \rho$	Gauss' law
$\oint_s \mathbf{B}\cdot d\mathbf{s} = 0$	$\nabla\cdot\mathbf{B} = 0$	nonexistence of magnetic monopoles
$\oint \mathbf{E}\cdot d\mathbf{l} = \int_s \left(\frac{-\partial\mathbf{B}}{\partial t}\right)\cdot d\mathbf{s}$	$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$	Faraday's law
$\oint \mathbf{H}\cdot d\mathbf{l} = \int_s \left(\mathbf{J}_c + \frac{\partial\mathbf{D}}{\partial t}\right)\cdot d\mathbf{s}$	$\nabla\times\mathbf{H} = \mathbf{J}_c + \frac{\partial\mathbf{D}}{dt}$	Ampère's law

Free-Space Form

integral form	point form
$\oint_s \mathbf{D}\cdot d\mathbf{s} = 0$	$\nabla\cdot\mathbf{D} = 0$
$\oint_s \mathbf{B}\cdot d\mathbf{s} = 0$	$\nabla\cdot\mathbf{B} = 0$
$\oint \mathbf{E}\cdot d\mathbf{l} = \int_s \left(\frac{-\partial\mathbf{B}}{\partial t}\right)\cdot d\mathbf{s}$	$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$
$\oint \mathbf{H}\cdot d\mathbf{l} = \int_s \left(\frac{\partial\mathbf{D}}{\partial t}\right)\cdot d\mathbf{s}$	$\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{dt}$

Electromagnetic Field Vector Equations

$$\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0(1+\chi_e)\mathbf{E}$$

$$\mathbf{B} = \mu\mathbf{H} = \mu_0\mathbf{H} + \mu_0\mathbf{M} = \mu_0(1+\chi_m)\mathbf{H}$$

$$\mathbf{J} = \sigma\mathbf{E} = \rho\mathrm{v}$$

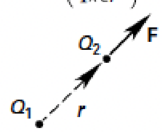
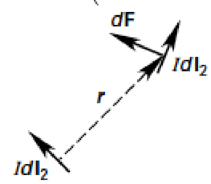
Appendix E: The Greek Alphabet

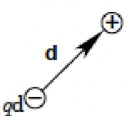
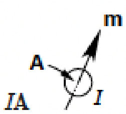
A	α	alpha	N	υ	nu
B	β	beta	Ξ	ξ	xi
Γ	γ	gamma	O	o	omicron
Δ	δ	delta	Π	π	pi
E	ε	epsilon	P	ρ	rho
Z	ζ	zeta	Σ	σ	sigma
H	η	eta	T	τ	tau
Θ	θ	theta	ϒ	υ	upsilon
I	ι	iota	Φ	ϕ	phi
K	κ	kappa	X	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
M	μ	mu	Ω	ω	omega

Appendix F: SI Prefixes

symbol	prefix	value
a	atto	10^{-18}
f	femto	10^{-15}
p	pico	10^{-12}
n	nano	10^{-9}
μ	micro	10^{-6}
m	milli	10^{-3}
c	centi	10^{-2}
d	deci	10^{-1}
da	deka	10
h	hecto	10^{2}
k	kilo	10^{3}
M	mega	10^{6}
G	giga	10^{9}
T	tera	10^{12}
P	peta	10^{15}
E	exa	10^{18}

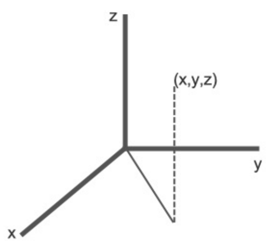
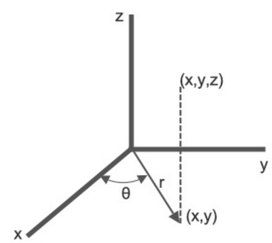
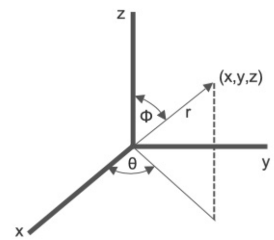
Appendix G: Comparison of Electric & Magnetic Equations

equation description	electric version	magnetic version	remarks
experimental force law	Coulomb's law $\mathbf{F} = \left(\frac{Q_1 Q_2}{4\pi\epsilon r^2}\right)\mathbf{r}$ 	force between two current elements $d\mathbf{F} = \left(\frac{\mu_0}{4\pi}\right)\left(\frac{I_2 d\mathbf{l}_2 \times (I_1 d\mathbf{l}_1 \times r)}{r^2}\right)$ 	The term Idl in the magnetic column is the equivalent of a "magnetic charge" q_m. The I or the dl can be the vector. The r is a unit vector pointing from 1 to 2.
field definitions from force law	$\mathbf{F} = Q\mathbf{E}$	$d\mathbf{F} = \mathbf{I}\times\mathbf{B}\,dl$ current element $d\mathbf{F} = \mathbf{J}\times\mathbf{B}\,dV$ distributed current element $d\mathbf{F} = q\,\mathbf{v}\times\mathbf{B}$ moving charge	The V used in this row represents volume, not voltage. The v is the velocity.
general force law	$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$ $d\mathbf{F} = (\rho\mathbf{E} + \mathbf{J}\times\mathbf{B})\,dV$ where $dQ = \rho\,dV$		The V in this row represents the volume, not voltage. The v is the velocity.
definition of scalar and vector potential	$E = -\nabla V$	$\mathbf{B} = \nabla\times\mathbf{A}$	A is the magnetic vector potential.
Poisson's equation for the potential function	$\nabla^2 V = -\frac{\rho}{\epsilon}$	$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$	From a knowledge of the charge distribution, the potential can be found and then the E and B fields determined.
Gauss's law enclosing charge and Ampère's law enclosing current	$\iint \mathbf{D}\cdot d\mathbf{A} = \iiint \rho\,dV = Q$ $\nabla\cdot\mathbf{D} = \rho$	$\oint H\cdot dl = I$ $\nabla\times\mathbf{H} = \mathbf{J}$	The V in this row represents volume.
constitutive relations	$\mathbf{D} = \epsilon\mathbf{E}$ $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$	$\mathbf{B} = \mu\mathbf{H}$ $\mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M}$	The second set of equations is always valid. The first set assumes the medium is linear and isotropic.
definitions of relative permittivity and permeability	$\epsilon_r = \frac{\epsilon}{\epsilon_0}$ $\epsilon_0 = 8.854\times10^{-12}$ F/m	$\mu_r = \frac{\mu}{\mu_0}$ $\mu_0 = 4\pi\times10^{-7}$ H/m	

equation description	electric version	magnetic version	remarks
capacitance and inductance of a field cell	$\epsilon_0 = \frac{C}{l}$	$\mu_0 = \frac{L}{l}$	Field cells are a construct designed to represent free space in terms of a parallel plate capacitor and an inductor. This capacitance and inductance exist regardless of the presence of an electric or magnetic field.
capacitance and inductance	$C = \frac{Q}{V}$	$L = \frac{\Lambda}{I}$	Λ is the flux linkage.
energy density of a field	$U = \frac{1}{2}\epsilon E^2$	$U = \frac{1}{2}\mu H^2$	Both energy and momentum are carried by a field.
energy stored by capacitance and inductance	$W = \frac{1}{2}CV^2$	$W = \frac{1}{2}LI^2$	
electromotive and magnetomotive force with sources present	$\oint \mathcal{E}\cdot d\mathbf{l} = \mathcal{E} = V$	$\oint \mathbf{H}\cdot d\mathbf{l} = NI = F_m = V_m$	The $\mathcal{E}$ is the emf, not the permittivity. Without sources present, both line integrals are equal to zero.
dipole moments	$\mathbf{p} = q\mathbf{d}$ 	$\mathbf{m} = I\mathbf{A}$ 	
dipole torque	$\mathbf{T} = \mathbf{p} \times \mathbf{E}$	$\mathbf{T} = \mathbf{m} \times \mathbf{B}$	This torque occurs due to the dipole being immersed in an external E or B field.
dipole potential energy	$W = -\mathbf{p}\cdot\mathbf{E}$	$W = -\mathbf{m}\cdot\mathbf{B}$	

electric	magnetic
emf $= V = IR$	mmf $= V_m = \phi\mathcal{R}$
current I	flux ϕ
emf $\mathcal{E}$ or V	mmf V_m
resistance $R = \rho l/A = l/\sigma A$	reluctance $\mathcal{R} = l/\mu A$
resistivity ρ	reluctivity $1/\mu$
conductance $G = 1/R$	permeance $P_m = \mu A/l$
conductivity $\sigma = 1/\rho$	permeability μ

Appendix H: Coordinate Systems and Related Operations

Mathematical Operations	Rectangular Coordinates	Cylindrical Coordinates	Spherical Coordinates
Conversion to Rectangular Coordinants	 $x = x$ $y = y$ $z = z$	 $x = r\cos\theta$ $y = r\sin\theta$ $z = z$	 $x = r\sin\phi\cos\theta$ $y = r\sin\phi\sin\theta$ $z = r\cos\phi$
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$	$\nabla f = \frac{\partial f}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{\theta} + \frac{\partial f}{\partial z}\mathbf{k}$	$\nabla f = \frac{\partial f}{\partial r}\mathbf{r} + \frac{1}{r}\frac{\partial f}{\partial \phi}\boldsymbol{\phi} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \theta}\boldsymbol{\theta}$
Divergence	$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$	$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial (A_\phi \sin\phi)}{\partial \phi} + \frac{1}{r\sin\phi}\frac{\partial A_\theta}{\partial \theta}$
Curl	$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$	$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{1}{r}\mathbf{r} & \boldsymbol{\theta} & \frac{1}{r}\mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & A_\theta & A_z \end{vmatrix}$	$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{1}{r^2\sin\theta}\mathbf{r} & \frac{1}{r^2\sin\theta}\boldsymbol{\phi} & \frac{1}{r}\boldsymbol{\theta} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ A_r & rA_\phi & rA\theta A_\phi \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\nabla^2 f = \frac{1}{r}\frac{\partial r}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}$	$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) + \frac{1}{r^2\sin^2\phi}\left(\frac{\partial^2 f}{\partial \theta^2}\right)$